



4- Linear Ordinary Differential Equations with Constant Coefficients

The general form:

$$\ddot{y} + P(x)\dot{y} + Q(x)y = R(x)$$

if $R(x) = 0$ $\therefore$ Homogenous Equations معادلة متجانسة

if $R(x) \neq 0$ $\therefore$ Non-Homogenous Equations معادلة غير متجانسة

where:

$P(x)$: The function adjacent to $(\dot{y})$ when the coefficient of $\dot{y}$ is equal to 1.

$P(x)$: هو الدالة المجاورة لل $\dot{y}$ عندما معامل $\dot{y}$ يساوي 1.

$Q(x)$: The function adjacent to (y) when the coefficient of y is equal to 1.

$Q(x)$: هو الدالة المجاورة لل y عندما معامل y يساوي 1.

$R(x)$: The right side function is free from (y) and its derivatives when the coefficient of $\dot{y}$ is equal to 1.

$R(x)$: هو الدالة في الجهة اليمنى والخالية من المتغير y ومشتقاته عندما معامل $\dot{y}$ يساوي 1.

4.1- Homogeneous Second Order Linear Differential Equations

Theorem: if y_1 and y_2 are two solutions to the homogenous equation, then $y_c = c_1 y_1 + c_2 y_2$ is general solution for Homogenous Equations, where c_1 and c_2 are constant.

إذا كان y_1 و y_2 حلين للمعادلة التفاضلية المتجانسة فإن $y_c = c_1 y_1 + c_2 y_2$ هو الحل العام لهذه المعادلة.
معادلة الحل العام هو:

$$y_c = c_1 y_1 + c_2 y_2$$

في حالة توفر احد الحلول (y_1) نستطيع ان نحصل على الحل الثاني (y_2) من خلال هذه المعادلة:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

وبالتالي يكون الحل العام لمثل هذه الحالة كما يلي:

$$\therefore y_c = c_1 y_1 + c_2 y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

Example (1): Find the general solution if $y_1 = x^2$ is a solution of the equation:

$$x^2 \ddot{y} + x \dot{y} - 4y = 0$$

Solve:

$$x^2 \ddot{y} + x \dot{y} - 4y = 0 \quad \div x^2$$

$$\ddot{y} + \frac{1}{x} \dot{y} - \frac{4}{x^2} y = 0$$

The general equation: $\ddot{y} + P(x)\dot{y} + Q(x)y = 0$

$$\therefore P(x) = \frac{1}{x}, \quad y_1 = x^2$$

$$\therefore y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$y_2 = x^2 \int \frac{e^{-\int \frac{1}{x} dx}}{x^4} dx = x^2 \int \frac{e^{-\ln x}}{x^4} dx = x^2 \int \frac{e^{\ln x^{-1}}}{x^4} dx$$

$$y_2 = x^2 \int x^{-1} \cdot x^{-4} dx = x^2 \int x^{-5} dx = x^2 \cdot \frac{x^{-4}}{-4} = x^2 \cdot \frac{-1}{4 x^4}$$

$$\therefore y_2 = \frac{-1}{4 x^2}$$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_c = c_1 x^2 + c_2 \frac{-1}{4 x^2}$$

$$\therefore y_c = c_1 x^2 - \frac{c_2}{4 x^2}$$

Example (2): Find the general solution for the equation $\dot{y} - 2y - 3y = 0$ if $y_1 = e^{3x}$?

Solve:

$$\dot{y} - 2y - 3y = 0$$

The general equation: $\dot{y} + P(x)y + Q(x)y = 0$

$$\therefore P(x) = -2, \quad y_1 = e^{3x}$$

$$\therefore y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$y_2 = e^{3x} \int \frac{e^{-\int -2 dx}}{(e^{3x})^2} dx = e^{3x} \int \frac{e^{2x}}{e^{6x}} dx = e^{3x} \int e^{2x} \cdot e^{-6x} dx$$

$$y_2 = e^{3x} \int e^{-4x} dx = e^{3x} \cdot \frac{e^{-4x}}{-4}$$

$$\therefore y_2 = \frac{-1}{4} e^{-x}$$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_c = c_1 e^{3x} + c_2 \frac{-1}{4} e^{-x}$$

$$\therefore y_c = c_1 e^{3x} - \frac{c_2}{4} e^{-x}$$

H.W: Using the given solution find a general solution of each of the following equation:

1) $\dot{y} + y = 0, \quad y_1 = \sin x$

Ans: $y_c = c_1 \sin x - c_2 \cos x$

2) $x^2 \dot{y} + (x^2 - 2x)y + (x + 2)y = 0, \quad y_1 = x$

Ans: $y_c = c_1 x - c_2 x e^{-x}$