Engineering mechanics

Lecture 10:

Moment of Inertia

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Objectives

After studying this Lecture, you will be able to

- Define Moment of Inertia
- Determine Moment of Inertia

1. Moment of Inertia

INTRODUCTION

We have already discussed that the moment of a force (P) about a point, is the product of the force and perpendicular distance (x) between the point and the line of action of the force (i.e. P.x).This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance (x) between the point and the line of action of the force i.e. P.x (x) = Px2, then this quantity is called moment of the moment of a force or second moment of force or moment of inertia (briefly written as M.I.). Sometimes, instead of force, area or mass of a figure or body is taken into consideration. Then the second moment is known as second moment of area

1. Moment of Inertia

UNITS OF MOMENT OF INERTIA

As a matter of fact the units of moment of inertia of a plane area depend upon the units of
the area and the length. e.g.,

1. If area is in m2 and the length is also in m, the moment of inertia is expressed in m4.
2. If area in mm2 and the length is also in mm, then moment of inertia is expressed in mm4.

METHODS FOR MOMENT OF INERTIA

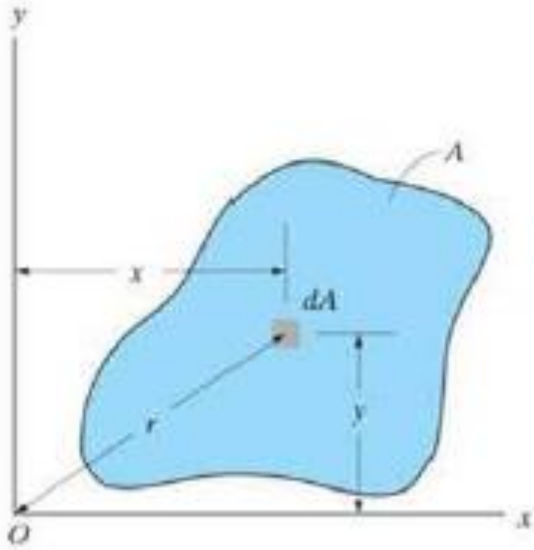
The moment of inertia of a plane area (or a body) may be found out by any one of the following
two methods :

1. By Routh's rule
2. By Integration.

1. Moment of Inertia

Moment of Inertia. By definition, the moments of inertia of a differential area dA about the x and y axes are $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively, Fig. 10–2. For the entire area A the *moments of inertia* are determined by integration; i.e.,

$$I_x = \int_A y^2\,dA$$
$$I_y = \int_A x^2\,dA$$



1. Polar Moment of Inertia

For the entire area the polar moment of inertia is:

$$J_O = \int_A r^2 \, dA = I_x + I_y$$

Parallel axis theorem for in area

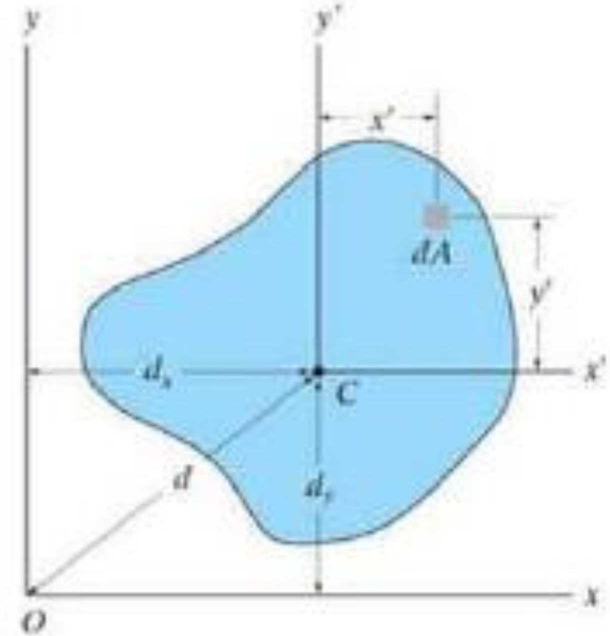
1. Moment of Inertia

$$I_x = \int_A (y' + d_y)^2 \, dA$$
$$= \int_A y'^2 \, dA + 2d_y \int_A y' \, dA + d_y^2 \int_A dA$$

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$I_y = \bar{I}_{y'} + Ad_x^2$$

$$J_O = \bar{J}_C + Ad^2$$



Radius of gyration

The *radius of gyration* of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are *known*, the radii of gyration are determined from the formulas

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

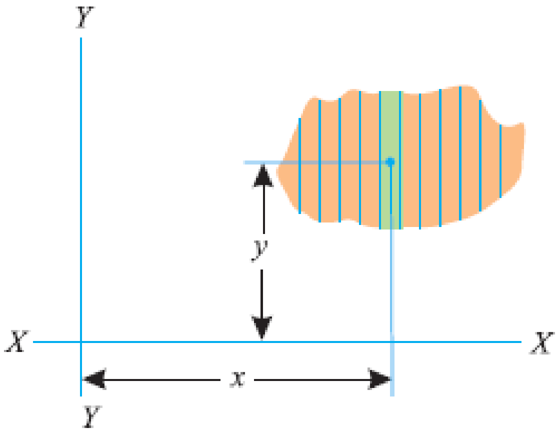
$$k_O = \sqrt{\frac{J_O}{A}}$$

1. Moment of Inertia

MOMENT OF INERTIA BY INTEGRATION

The moment of inertia of an area may also be found out by the method of integration as discussed below:

Consider a plane figure, whose moment of inertia is required to be found out about *X-X* axis and *Y-Y* axis as shown in Fig . Let us divide the whole area into a no. of strips. Consider one of these strips.



Moment of inertia by integration.

Let dA = Area of the strip

x = Distance of the centre of gravity of the strip on *X-X* axis and

y = Distance of the centre of gravity of the strip on *Y-Y* axis.

We know that the moment of inertia of the strip about *Y-Y* axis

$$= dA \cdot x^2$$

Now the moment of inertia of the whole area may be found out by integrating above equation. *i.e.*,

$$I_{YY} = \sum dA \cdot x^2$$

Similarly $I_{XX} = \sum dA \cdot y^2$

In the following pages, we shall discuss the applications of this method for finding out the moment of inertia of various cross-sections.

1. Moment of Inertia

MOMENT OF INERTIA OF A RECTANGULAR SECTION

Consider a rectangular section *ABCD* as shown in Fig. whose moment of inertia is required to be found out.

Let b = Width of the section and

d = Depth of the section.

Now consider a strip *PQ* of thickness dy parallel to *X-X* axis and at a distance y from it as shown in the figure

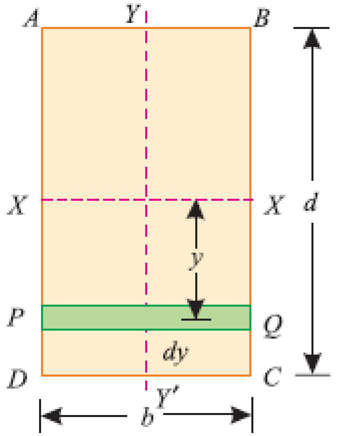
∴ Area of the strip

$= b.dy$

We know that moment of inertia of the strip about *X-X* axis,

$= \text{Area} \times y^2 = (b.\, dy)\, y^2 = b.\, y^2.\, dy$

Now *moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina *i.e.* from $-\frac{d}{2}$ to $+\frac{d}{2}$,



Rectangular section.

* This may also be obtained by Routh's rule as discussed below :

$$I_{xx} = \frac{AS}{3} \quad \text{...(for rectangular section)}$$

where area, $A = b \times d$ and sum of the square of semi axes *Y-Y* and *Z-Z*,

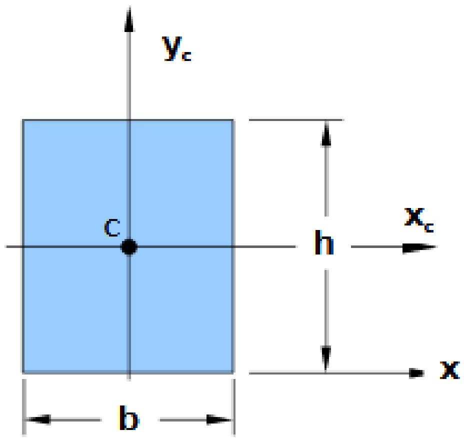
$$S = \left(\frac{d}{2}\right)^2 + 0 = \frac{d^2}{4}$$

$$\therefore \quad I_{xx} = \frac{AS}{3} = \frac{(b \times d) \times \frac{d^2}{4}}{3} = \frac{bd^3}{12}$$

1. Moment of Inertia

Example 4:

Given the moment of inertia of a rectangle about its centroidal axis, apply the parallel axis theorem to find the moment of inertia for a rectangle about its base.



$$I_{xc} = \frac{b \cdot h^3}{12}$$

$$distance = \frac{h}{2} \quad area = b \cdot h$$

$$I_x = \frac{b \cdot h^3}{12} + \left(\frac{h}{2}\right)^2 \cdot (b \cdot h)$$

$$= \frac{b \cdot h^3}{12} + \frac{b \cdot h^3}{4}$$

$$= \frac{b \cdot h^3}{12} + \frac{3 \cdot b \cdot h^3}{12} = \frac{b \cdot h^3}{3}$$

1. Moment of Inertia

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the $x'-y'$ plane and passing through the centroid C.

SOLUTION (CASE 1)

Part (a). The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

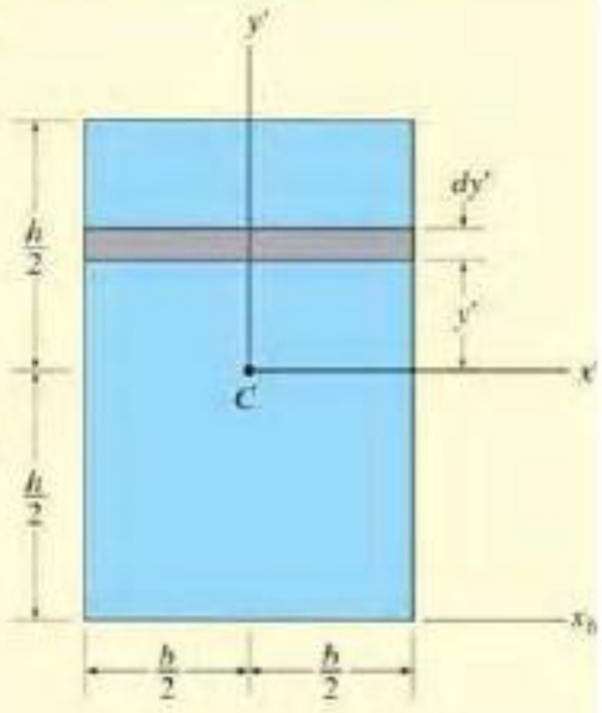


Fig. 10–5

1. Moment of Inertia

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \qquad \textit{Ans.}$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10–2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad \textit{Ans.}$$

1. Moment of Inertia

Example *Find the moment of inertia of a rectangular section 30 mm wide and 40 mm deep about X-X axis and Y-Y axis.*

Solution. Given: Width of the section (b) = 30 mm and depth of the section (d) = 40 mm.

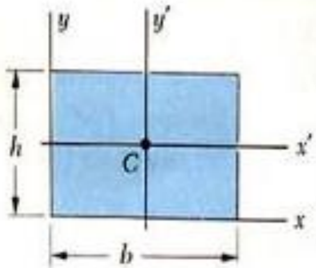
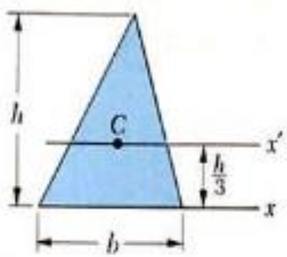
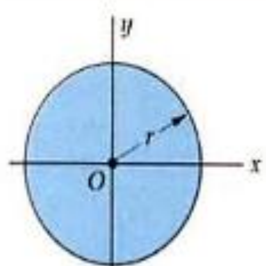
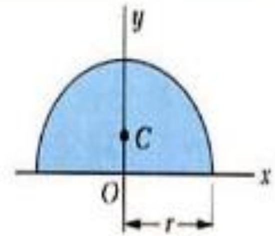
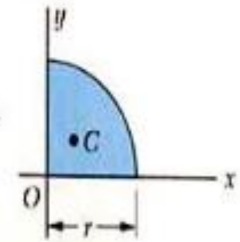
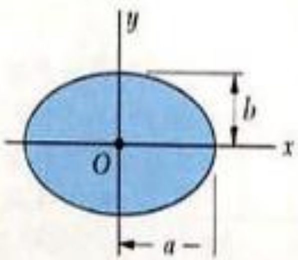
We know that moment of inertia of the section about an axis passing through its centre of gravity and parallel to X-X axis,

$$I_{XX} = \frac{bd^3}{12} = \frac{30 \times (40)^3}{12} = 160 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

Similarly

$$I_{YY} = \frac{db^3}{12} = \frac{40 \times (30)^3}{12} = 90 \times 10^3 \text{ mm}^4 \quad \textbf{Ans.}$$

1. Moment of Inertia

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

1. Moment of Inertia

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the *x* axis.

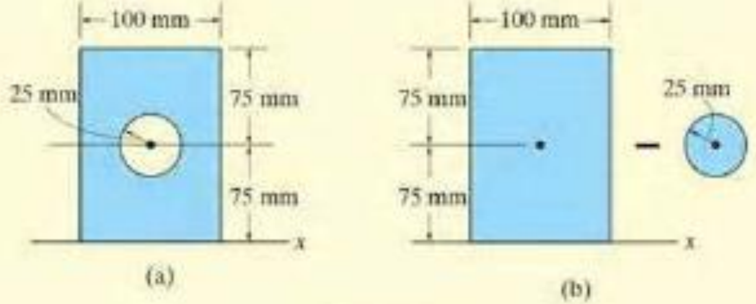


Fig. 10–8

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the *x* axis are determined using the parallel-axis theorem and the data in the table on the inside back cover.

1. Moment of Inertia

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\text{ mm}^4$$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\text{ mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$

1. Moment of Inertia

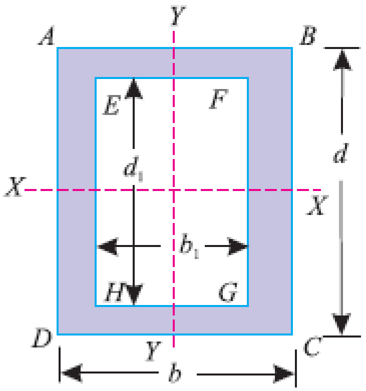
MOMENT OF INERTIA OF A HOLLOW RECTANGULAR SECTION

Consider a hollow rectangular section, in which *ABCD* is the main section and *EFGH* is the cut out section as shown in Fig

Let b = Breadth of the outer rectangle,

d = Depth of the outer rectangle and

b_1, d_1 = Corresponding values for the cut out rectangle.



Hollow rectangular section.

We know that the moment of inertia, of the outer rectangle *ABCD* about *X-X* axis

$$= \frac{bd^3}{12} \quad \text{...}(i)$$

and moment of inertia of the cut out rectangle *EFGH* about *X-X* axis

$$= \frac{b_1 d_1^3}{12} \quad \text{...}(ii)$$

∴ M.I. of the hollow rectangular section about *X-X* axis,

I_{XX} = M.I. of rectangle *ABCD* – M.I. of rectangle *EFGH*

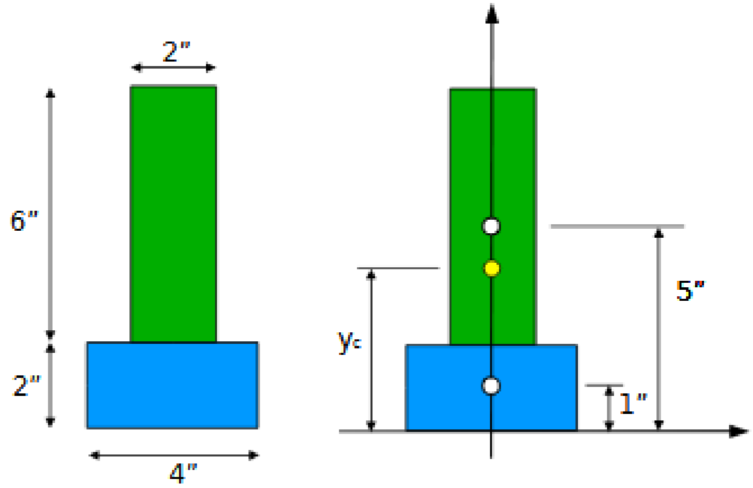
$$= \frac{bd^3}{12} - \frac{b_1 d_1^3}{12}$$

Similarly, $$I_{yy} = \frac{db^3}{12} - \frac{d_1 b_1^3}{12}$$

Note : This relation holds good only if the centre of gravity of the main section as well as that of the cut out section coincide with each other.

1. Moment of Inertia

- Example 5:
  Find the centroidal moment of inertia for a T-shaped area.
  1) First, locate the centroid of each rectangular area relative to a common base axis, then...
  2) ...determine the location of the centroid of the composite.



$$x_c = 0 \qquad \text{(by symmetry)}$$

$$y_c = \frac{\Sigma\, y_i A_i}{\Sigma\, A_i} = \frac{(5\cdot(6\cdot 2)+1\cdot(2\cdot 4))}{(2\cdot 4+6\cdot 2)} = \frac{(60+8)}{20} = 3.4 \text{ in}$$

1. Moment of Inertia

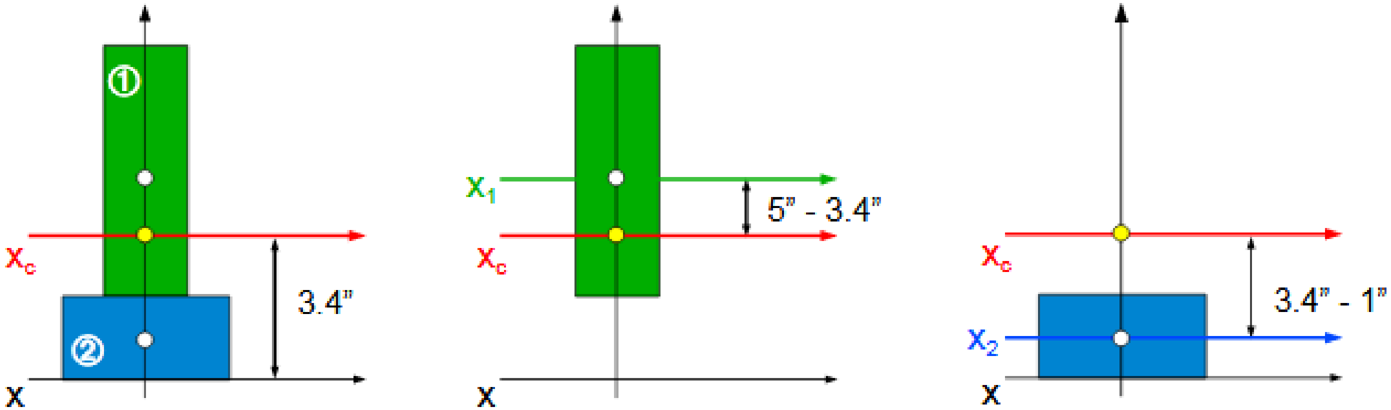
3) Find centroidal moment of inertia about the x-axis

$$I_{xc} = I_{x1} + I_{x2} \quad \text{where} \quad I_{xi} = \frac{b_i \cdot h_i^3}{12} + d_i^2 \cdot (b_i \cdot h_i)$$

$$I_{x1} = \frac{2 \cdot 6^3}{12} + (5 - 3.4)^2 \cdot 2 \cdot 6 = 66.72 \cdot \text{in}^4$$

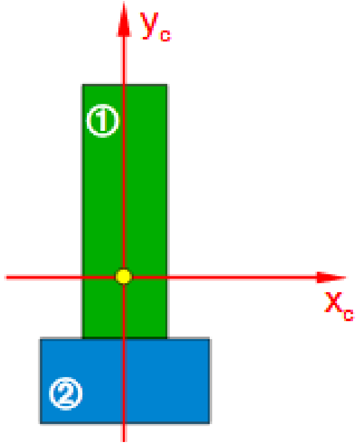
$$I_{x2} = \frac{4 \cdot 2^3}{12} + (3.4 - 1)^2 \cdot 2 \cdot 4 = 48.75 \cdot \text{in}^4$$

$$I_{xc} = I_{x1} + I_{x2} = 66.72 + 48.75 = 115.46 \text{ in}^4$$



1. Moment of Inertia

4) Find the centroidal moment of inertia about the y-axis. Since the centroidal y-axis for each shape and for the composite is coincident, the moments of inertia are additive.



$$I_{yc} = I_{y1} + I_{y2}$$

$$I_{yi} = \frac{h_i \cdot b_i^3}{12}$$

$$I_{yc} = \frac{6 \cdot 2^3}{12} + \frac{2 \cdot 4^3}{12} = 14.66 \cdot \text{in}^4$$

1. Moment of Inertia

Example

Find Moment of Inertia of this object:

- First we divide the object into two standard shapes present in the reference tables, the find the MI for each respective shape.