Complex numbers

The Complex numbers

Def: The order pair z=(x, y) where x and y are real numbers is called the complex number.

Notations:

1. The complex number (0,y) is called pure imaginary number.
2. The real number x is called the real part of z and The real number y is called the imaginary part of z.
3. We say that the complex numbers(x_1, y_1) and(x_2, y_2) are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.
4. The add ion and the multiplication are defined as: $z_1 + z_{2=} (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

   $z_1 . z_{2=}(x_1, y_1). (x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$

1. If i= $\sqrt{-1}$ then we can write the complex number z = x+iy and
   $z_1 . z_{2=}(x_1, y_1). (x_2, y_2) = (x_1x_2 - y_1y_2 , x_1y_2 + x_2y_1)$

   $z_1 + z_{2=}(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Examples:

1. (2+3i)+(1+4i)=(2+1)+ (3+4)i=3+7i
2. (1−i).(2+3i)=(1×2 –(-1× 3) $+ (1 \times 3 + (2 \times (-1))$i=5+i
3. $i^3 = (i^2)$ i=−i

Algebraic properties :

1.The commutative law : $z_1 + z_{2=}z_2 + z_1$ and $z_1 . z_{2=} \; z_2 . z_1$ $\forall \; z_1, z_2 \in ₵$

2. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

   $z_1 . (z_2 . z_3) = (z_1 . z_2) . z_3$

4.The additive identity is 0=0+0i then $\forall z\epsilon ₵$,z+0=0+z=z

5.The multiple identity is 1=1+0i then $\forall z\epsilon ₵$,z.1=1.z=z

6.The additive inverse $\forall z\epsilon ₵ \ \exists -z=-x-\mathrm{i}y$ st .z+(-z)= −z+ z=0

7. The multiplicative inverse $\forall z\epsilon ₵ \ \exists z^{-1}$ s t. $z.z^{-1}=z^{-1}.z=1$

8. the conjugate of the complex number z=x+ i y is z^{-} = x− i y

Examples:

1.(6+5i)-(4-3i)+(2+7i)=4+15i

3.($\sqrt{2}$-i)−i(1-$\sqrt{2}$ i)=($\sqrt{2}$-i)−(i + $\sqrt{2}$)=-2i

4The conjugate of 3-7i= 3+7i

6. find the inverse of −2+3i

$$z^{-1}=\frac{1}{z}=\frac{1}{-2+3i}=\frac{1}{-2+3i}\times\frac{-2-3i}{-2-3i}=\frac{-2-3i}{13}=\frac{-2}{13}+\frac{-3i}{13}$$

Graphical representation of the complex number:

Every complex number z=x+ iy corresponding one point in the plane XY For example (0,0) corresponds to the complex number z=0+0i and the number z represents the distance from (0,0) to (x ,y) therefore the plane is called the complex plane ,X is called the real axis and Y is called the imaginary axis. Y

The absolute value of the complex number:

The absolute value of the complex number z=x+ i y is defined as fellows:

$$|z| = \sqrt{x^2 + y^2}$$

Note:

1. The number z represents the distains between the origin and (x, y)
2. If $z_1 = (x_1, y_1)$and $z_2 = (x_2, y_2)$ then the distance between them is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3.$|z_1.z_2| = |z_1|.|z_2|$

4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $\qquad z_2 \neq 0$

Example:

z= 2-3i then

$|z| = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$

polar form the complex number:

Let r, θ are the polar coordinates correponding to (x, y)that represents z

x=r cos θ , $y = \text{r} \sin \theta$

z=r(cos θ + isinθ) $= re^{i\theta}$

s .t . $r = |z| = \sqrt{x^2 + y^2}$ and θ it is the angle of the complex number z

, it is called (argument)andcan be write $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Examples:

1. write z=1+i by the polar form :

sol:

r= $\sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2 = \sqrt{2}}$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = e^{i\frac{\pi}{4}}$$

1. $z = i$

$$r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

seince $\cos\theta$ and $\sin\theta$ are pirodic in 2π then $\arg z = \theta + 2k\pi$

if k=0, $\arg z = \theta$, $-\pi < \theta < \pi$

notations:

1. $\arg(z_1 + z_2) = \arg z_1 + \arg z_1$
2. let $z_1 = r(\cos\theta + i\sin\theta)$, $z_2 = p(\cos\emptyset + i\sin\emptyset)$

   $z_1 . z_2 = rp(\cos\theta + i\sin\theta).(\cos\emptyset + i\sin\emptyset) = rp(\cos\theta\cos\emptyset + i\sin\theta\sin\emptyset) = rp(\cos(\theta$

$\arg(z_1 . z_2) = \theta + \emptyset$

1. $\arg \frac{z_1}{z_2} = arg\ z_1 - arg z_2$
2. for all the integer number n $z^n = r^n(\cos n\ \theta + isin\ n\theta) = r^n(\cos\theta + i\sin\theta)^n$

Example: represent the following complex numbers in the standard form:

z=$e^{i\theta}$ $\rightarrow$ $r = 1\ \ and\ \theta = \frac{\pi}{2}$

X=r cos$\theta = 1 * cos\frac{\pi}{2} = 0$ $y = r\ sin\theta$=1.sin$\frac{\pi}{2} = 1 * 1 = 1$

Z=x+iy=0+i.

The complex function:

Let S be non empty set of the points in the complex plane if ∀ $z \epsilon S$ ∃ w

s.t. w=f(z). i.e. f: S→ ₵ , S is called domain f and f(z) is called the range.

We can write f(z) by the following:

w=f(z) =u(x ,y)+iv(x ,y) ,u ,v are real functions.

Example:

1. f(z)=$x^2 + 2y - \mathrm{i}2xy^3$,, u(x, y)= $x^2 + 2y$,v$(x, y) = -\mathrm{i}2xy^3$
2. f(z)= z^2 write f(z) by u and v.

sol: z=x +i y → f(z)=$(x + iy^2)$

f(z)= $x^2 + 2\mathrm{i}xy - y^2$ =$x^2 + y^2 - 2\mathrm{i}xy$

u(x, y)= $x^2 + y^2$, v$(x, y) = -2\mathrm{i}xy$