جامعة المستقبل
AL MUSTAQBAL UNIVERSITY
كلية العلوم

1^{st} class

2024- 2025

Mathematics 1

Lecture 7

Asst. Lect. Mohammed Jabbar
mohammed.jabbar.obaid@uomus.edu.iq

الرياضيات الاساسية: المرحلة الاولى

مادة الرياضيات 1

المحاضرة السابعة

استاذ المادة: م.م محمد جبار

Cybersecurity Department
قسم الأمن السيبراني

Contents

1 Sequence and Series

1.1 Sequence

Definition 1.1. A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ represents the set of natural numbers and $\mathbb{R}$ represents the set of real numbers. The sequence is denoted as $\{a_n\}$, where a_n is the value of the function at $n \in \mathbb{N}$, referred to as the n-th term of the sequence.

In expanded form, a sequence can be written as:

$$\{a_n\} = \{a_1, a_2, a_3, \dots, a_n, \dots\}.$$

Example 1.1. Some type of sequences

1. A constant sequence: $a_n = c$ for all n, e.g., $\{5, 5, 5, \dots\}$.
2. A linear sequence: $a_n = 2n + 1$, e.g., $\{3, 5, 7, 9, \dots\}$.
3. A geometric sequence: $a_n = 3 \cdot 2^{n-1}$, e.g., $\{3, 6, 12, 24, \dots\}$.

Example 1.2. The explicit values of the sequences from are as follows:

(a) $\{a_n\} = \{\frac{1}{n}\}$

Explicit values:

$$\{a_n\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}.$$

(b) $\{a_n\} = \{\sqrt{n}\}$

Explicit values:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\} = \{1, \sqrt{2}, \sqrt{3}, 2, \dots\}.$$

(c) $\{a_n\} = \left\{\frac{1}{n!}\right\}$

Explicit values:

$$\{a_n\} = \left\{\frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots\right\}.$$

(d) $\{a_n\} = \{\sin n\}$

Explicit values:

$$\{a_n\} = \{\sin 1, \sin 2, \sin 3, \sin 4, \dots\}.$$

(Since $\sin n$ depends on the angle n in radians, the exact numeric values can be computed as needed.)

(e) $\{a_n\} = \left\{\frac{e^n}{n!}\right\}$

Explicit values:

$$\{a_n\} = \left\{\frac{e^1}{1!}, \frac{e^2}{2!}, \frac{e^3}{3!}, \frac{e^4}{4!}, \dots\right\}.$$

Approximate values for initial terms:

$$\{a_n\} \approx \{e, \frac{e^2}{2}, \frac{e^3}{6}, \frac{e^4}{24}, \dots\}.$$

(f) $\{a_n\} = \left\{\frac{n-1}{n}\right\}$

Explicit values:

$$\{a_n\} = \left\{\frac{1-1}{1}, \frac{2-1}{2}, \frac{3-1}{3}, \frac{4-1}{4}, \dots\right\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}.$$

Example 1.3. Finding the General Term of the Sequence Consider the sequence:

$$\{-1, 1, -1, 1, -1, 1, \dots\}$$

Solution:

We observe the following values for the first few terms:

$$a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = 1, \ldots$$

This sequence alternates between -1 and 1. Hence, the general term of the sequence can be expressed as:

$$a_n = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

A more concise way to write this term is:

$$a_n = (-1)^n$$

1.2 Series

Definition 1.2. A **series** is the sum of the terms of a sequence. If $\{a_n\}$ is a sequence, the series is given by:

$$S = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{n=1}^{\infty} a_n$$

Definition 1.3 (Power series)**.** If $\{a_n\}$ is a sequence of constants, the expression:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

is called a power series in x.

1.2.1 Convergence of a Series

A series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of its partial sums $\{S_n\}$ has a finite limit as $n \to \infty$. That is:

$$\lim_{n\to\infty} S_n = S \quad \text{(finite)}$$

where

$$S_n = \sum_{k=1}^{n} a_k$$

If $\lim_{n\to\infty} S_n$ does not exist or is infinite, the series diverges.

Examples of Series

1. **Geometric Series:**

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1 \\ \text{diverges}, & \text{if } |r| \geq 1 \end{cases}$$

2. **Harmonic Series:**

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

3. p**-Series:**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \leq 1 \end{cases}$$

Tests for Convergence

1. **Comparison Test:** If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.

2. **Ratio Test:** For a series $\sum a_n$, define:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1,$ the series converges. If $L > 1,$ the series diverges.

2 Taylor and Maclaurin Series

Taylor Series

If a function f can be represented by a power series in $(x-a)$, it is called Taylor's series and has the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

Example 2.1. Find Taylor series expansion of $\cos x$ about a point $a = 2\pi$.

Sol.

$$f(x) = \cos x, \quad f(2\pi) = \cos(2\pi) = 1$$

$$f'(x) = -\sin x, \quad f'(2\pi) = -\sin 2\pi = 0$$

$$f''(x) = -\cos x, \quad f''(2\pi) = -\cos 2\pi = -1$$

$$f'''(x) = \sin x, \quad f'''(2\pi) = \sin 2\pi = 0$$

$$f^{(iv)}(x) = \cos x, \quad f^{(iv)}(2\pi) = \cos 2\pi = 1$$

$$\cos x = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \frac{(x-2\pi)^6}{6!} + \cdots$$

Final Function:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-2\pi)^{2n}}{(2n)!}$$

□

Maclaurin series

When $a = 0$, Taylor's series is called the Maclaurin series.

Example 2.2. Find Maclaurin series for the function $f(x) = e^x$.

Sol.

$$f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = e^0 = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Final Function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

□

Example 2.3. Find the Maclaurin series expansion for $\sin x$.

Sol. Let $f(x) = \sin x$. At $x = 0$, we calculate the derivatives:

$$f(x) = \sin x, \quad f(0) = 0$$

$$f'(x) = \cos x, \quad f'(0) = 1$$

$$f''(x) = -\sin x, \quad f''(0) = 0$$

$$f'''(x) = -\cos x, \quad f'''(0) = -1$$

The Maclaurin series expansion is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Final Function:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

□

Function, $f(x)$	**Maclaurin Series (**$b = 0$**)**	**Taylor Series (**$b \neq 0$**)**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\sum_{n=0}^{\infty} \frac{e^b(x-b)^n}{n!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(b)(x-b)^{2n+1}}{(2n+1)!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$\sum_{n=0}^{\infty} \frac{f^{(2n)}(b)(x-b)^{2n}}{(2n)!}$
$\ln(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	-
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\sum_{n=0}^{\infty} \frac{(-1)^n (x-b)^n}{(1+b)^{n+1}}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$\sum_{n=0}^{\infty} \frac{(x-b)^n}{(1-b)^{n+1}}$
$\sinh(x)$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	-
$\cosh(x)$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	-
$\ln(x)$	$\ln(b) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-b)^n}{nb^n}$	Not defined for Maclaurin series

Table 1: Expanded Table of Taylor and Maclaurin Series for Common Functions