Derivatives

The derivative of a function represents the rate of change of one variable with respect to another variable.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Differentiation Rules:

1- Derivative of a Constant Function

If f has the constant value $f(x) = c$, then,

$$\frac{dy}{dx} = \frac{d}{dx}(c) = 0$$

EXAMPLE: Find the derivative of (a) *f(x)=7* b) *f(x)=-32* (c) *f(x)=4/7*

Solution:

(a) $\frac{d}{dx}(7) = 0$

(b) $\frac{d}{dx}(-32) = 0$

(c) $\frac{d}{dx}(4/7) = 0$

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2- Derivative of a Positive Integer Power

If n is a positive integer, then,

$$\frac{d}{dx}x^n = nx^{n-1}$$

EXAMPLE Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\frac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Solution

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

3- Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then,

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

EXAMPLE:

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}$$

4- Derivative Sum Rule

If u and y are differentiable functions of x, then their sum $u + y$ is differentiable at every point where u and y are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

EXAMPLE: If $y=x^4+12x$, find dy/dx

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12$$

EXAMPLE Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \qquad \text{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

5- Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

6- Derivative of the Natural Log Function

$$\frac{d}{dx}(lnx) = \frac{1}{x}$$

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7- Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

EXAMPLE Find the derivative of **(a)** $y = \frac{1}{x}\left(x^2 + e^x\right)$, **(b)** $y = e^{2x}$.

Solution

(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + (x^2 + e^x)\left(-\frac{1}{x^2}\right)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + (x - 1)\frac{e^x}{x^2}.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(b) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x) = 2e^x \cdot e^x = 2e^{2x}$

EXAMPLE Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

8- Derivative Quotient Rule

If u and v are differentiable at x and if $u(x) \neq 0$, then the quotient u/v is differentiable at x, and,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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EXAMPLE Find the derivative of **(a)** $y = \dfrac{t^2 - 1}{t^3 + 1}$, **(b)** $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

9- Derivatives of Trigonometric Functions

A. $\dfrac{d}{dx}(\sin x) = \cos x$

B. $\dfrac{d}{dx}(\cos x) = -\sin x$

C. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

D. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

E. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

F. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

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10- Derivatives of Inverse Trigonometric Functions, Hyperbolic Functions, and Inverse Hyperbolic Functions

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{1+x^2}}$$

11. The Chain Rule

A. Definition: Suppose that $f \circ g$ is the composite of two differentiable function $y=f(u)$ and $u=g(x)$. Then $f \circ g$ is a differentiable function of x whose derivative at each value of x is:

$$\text{or } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

B. Generalized Formulas: Let u be a differentiable function of x,

1. $\frac{d}{dx}(u^r) = r u^{r-1}\frac{du}{dx}$

2. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

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3. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

4. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

5. $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$

6. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$

7. $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

EXAMPLE The function

$$y = (3x^2 + 1)^2$$

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \qquad \text{Substitute for } u \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula $(3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$ gives the same result:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x. \end{aligned}$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule ·

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EXAMPLE Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot (\underbrace{2x + e^x}_{\text{derivative of the inside}}).$$

EXAMPLE Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x.$$

we see that the Chain Rule gives the formula

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

EXAMPLE

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2}\cdot\frac{d}{dx}(x^2) = 2xe^{x^2}.$$

EXAMPLE Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\
&= \sec^2(5 - \sin 2t)\cdot\frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\
&= \sec^2(5 - \sin 2t)\cdot\left(0 - \cos 2t\cdot\frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\
&= \sec^2(5 - \sin 2t)\cdot(-\cos 2t)\cdot 2 \\
&= -2(\cos 2t)\sec^2(5 - \sin 2t).
\end{aligned}$$

EXAMPLE

$$\begin{aligned}
\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6\frac{d}{dx}(5x^3 - x^4) && \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7 \\
&= 7(5x^3 - x^4)^6(5\cdot 3x^2 - 4x^3) \\
&= 7(5x^3 - x^4)^6(15x^2 - 4x^3)
\end{aligned}$$

EXAMPLE

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1} \\ &= -1(3x-2)^{-2}\frac{d}{dx}(3x-2) \qquad \text{Power Chain Rule with } u = 3x-2,\ n=-1 \\ &= -1(3x-2)^{-2}(3) \\ &= -\frac{3}{(3x-2)^2}\end{aligned}$$

EXAMPLE

$$\begin{aligned}\frac{d}{dx}(\sin^5 x) &= 5\sin^4 x \cdot \frac{d}{dx}\sin x \qquad \text{Power Chain Rule with } u=\sin x,\ n=5, \text{ because } \sin^n x \text{ means } (\sin x)^n,\ n \neq -1. \\ &= 5\sin^4 x\cos x\end{aligned}$$

EXAMPLE

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$

12. Implicit Differentiation

A. Procedure: Given an equation involving *x* and *y*, and assuming *y* is a differentiable function of *x*, we can find $\frac{dy}{dx}$ as follows:

1. Differentiate both sides of the equation with respect to x.

2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation, and move all other terms to the right side of the equation.

3. Factor $\frac{dy}{dx}$ out of the left side of the equation.

4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left-hand factor that does not contain $\frac{dy}{dx}$.

EXAMPLE: Find dy/dx if $y^2=x$

Solution:

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

EXAMPLE

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Differentiate both sides with respect to x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

Treat xy as a product and y as a function of x.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for dy/dx.

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Derivatives of Higher Order

EXAMPLE: If $y=3x^2+15x-3$, find d^2y/dx^2

Solution:

$$\frac{dy}{dx} = 6x + 15$$

$$\frac{d^2y}{dx^2} = 6$$

EXAMPLE: If $y=2/x^2$, find d^2y/dx^2

Solution: $\frac{dy}{dx} = 2(-2x^{-3}) = -4x^{-3}$

$$\frac{d^2y}{dx^2} = -4(-3x^{-4}) = 12x^{-4} = \frac{12}{x^4}$$

EXAMPLE: If $y = \sin^2 2x$, find d^2y/dx^2

Solution:

$$\frac{dy}{dx} = 2sin2x(cos2x).(2) = 4sin2xcos2x$$

$$\frac{d^2y}{dx^2} = 4[sin2x(-2sin2x) + cos2x(2cos2x)] = -8\sin^2 2x\, cos2x + 8\cos^2 2x$$

EXAMPLE: If $y = lnx^2$, find d^2y/dx^2

Solution:

$$\frac{dy}{dx} = \frac{1}{x}.2x = \frac{2x}{x^2} = \frac{2}{x}$$

$$\frac{d^2y}{dx^2} = 2(-1.x^{-1-1}) = -2x^{-2} = \frac{-2}{x^2}$$

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