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Mathematics 1

Lecture 9 and 10

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الرياضيات الاساسية: المرحلة الاولى

مادة الرياضيات 1

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1 Laplace Transformation

The Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \text{where } s \in \mathbb{C}.$$

Example 1.1 (Laplace Transform of a Constant Function)**.** Find the Laplace transform of the constant function $f(t) = c$.

Sol.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st} c\, dt.$$

Since c is a constant, it can be factored out:

$$\mathcal{L}\{c\} = c \int_0^\infty e^{-st}\, dt.$$

$$\mathcal{L}\{c\} = c \left[\frac{e^{-st}}{-s} \right]_0^\infty.$$

Since $e^{-s(\infty)} = 0$ and $e^0 = 1$, we get:

$$\mathcal{L}\{c\} = c \left[0 - \left(-\frac{1}{s} \right) \right] = c \cdot \frac{1}{s}.$$

Thus, the Laplace transform of a constant function is:

$$\mathcal{L}\{c\} = \frac{c}{s}, \quad s > 0.$$

□

Example 1.2. Find the Laplace transform of $f(t) = e^{at}$, a is constant.

Sol.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at}\, dt. \\ &= \int_0^\infty e^{-st+at}\, dt. \\ &= \int_0^\infty e^{-t(s-a)}\, dt.\end{aligned}$$

Using the standard integral formula:

$$\int e^{-bt}\, dt = \frac{e^{-bt}}{-b},$$

we substitute $b = s - a$ and evaluate from 0 to ∞:

$$\mathcal{L}\{e^{at}\} = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty .$$

Since $e^{-(s-a)\infty} = 0$ and $e^0 = 1$, we obtain:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \text{for } s > a.$$

□

1.1 Basic Laplace Transforms

1. **Linearity:**

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

2. **Time Shifting:**

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

3. **Frequency Shifting:**

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
a	$\frac{a}{s}, \quad s > 0$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
te^{at}	$\frac{1}{(s-a)^2}, \quad s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$

Example 1.3. Find the Laplace transform of $f(t) = e^t \cos(2t)$, a, b constants.

Sol. Using the standard formula:

$$\mathcal{L}\{e^{at} \cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a.$$

For $f(t) = e^t \cos(2t)$, we set $a = 1$ and $b = 2$, so:

$$\mathcal{L}\{e^t \cos(2t)\} = \frac{s-1}{(s-1)^2 + 4}.$$

Thus, the final result is:

$$\mathcal{L}\{e^t \cos(2t)\} = \frac{s-1}{(s-1)^2 + 4}, \quad s > 1.$$

□

Example 1.4. Find the Laplace transform of $f(t) = e^t t$, a, b constants.

Sol. Using the standard formula:

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}, \quad s > a.$$

For $f(t) = e^t t$, we set $a = 1$, so:

$$\mathcal{L}\{te^{t}\} = \frac{1}{(s-1)^2}, \quad s > 1.$$

Thus, the final result is:

$$\mathcal{L}\{te^{t}\} = \frac{1}{(s-1)^2}, \quad s > 1.$$

□

Example 1.5. Find the Laplace transform of $f(t) = e^{-2t} + t^3 - 4$, a, b constants.

Sol.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-2t}\} + \mathcal{L}\{t^3\} + \mathcal{L}\{-4\}$$

The Laplace transforms of each term are:

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}, \quad \mathcal{L}\{t^3\} = \frac{6}{s^4}, \quad \mathcal{L}\{-4\} = -\frac{4}{s}$$

Thus, the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \frac{1}{s+2} + \frac{6}{s^4} - \frac{4}{s}$$

□

1.2 Derivative Laplace transform

The Laplace transform of the derivatives of a function $f(t)$ can be computed using the following properties:

For the first derivative:

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

For the second derivative:

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \mathcal{L}\{f(t)\} - s \cdot f(0) - f'(0)$$

For the third derivative:

$$\mathcal{L}\{f^{(3)}(t)\} = s^3 \cdot \mathcal{L}\{f(t)\} - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$

For the fourth derivative:

$$\mathcal{L}\{f^{(4)}(t)\} = s^4 \cdot \mathcal{L}\{f(t)\} - s^3 \cdot f(0) - s^2 \cdot f'(0) - s \cdot f''(0) - f^{(3)}(0)$$

For the fifth derivative:

$$\mathcal{L}\{f^{(5)}(t)\} = s^5 \cdot \mathcal{L}\{f(t)\} - s^4 \cdot f(0) - s^3 \cdot f'(0) - s^2 \cdot f''(0) - s \cdot f^{(3)}(0) - f^{(4)}(0)$$

First Derivative

The first derivative of $f(t)$ is:

$$f'(t) = ae^{at}$$

The Laplace transform of $f'(t)$ is:

$$\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

Since $f(t) = e^{at}$ and $f(0) = 1$, we have:

$$\mathcal{L}\left\{f'(t)\right\} = s \cdot \frac{1}{s-a} - 1 = \frac{s}{s-a} - 1$$

Second Derivative:

The second derivative of $f(t)$ is:

$$f''(t) = a^2 e^{at}$$

The Laplace transform of $f''(t)$ is:

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \mathcal{L}\{f(t)\} - s \cdot f(0) - f'(0)$$

Since $f(0) = 1$ and $f'(0) = a$, we have:

$$\mathcal{L}\{f''(t)\} = s^2 \cdot \frac{1}{s-a} - s \cdot 1 - a = \frac{s^2}{s-a} - s - a$$

Third Derivative:

The third derivative of $f(t)$ is:

$$f^{(3)}(t) = a^3 e^{at}$$

The Laplace transform of $f^{(3)}(t)$ is:

$$\mathcal{L}\{f^{(3)}(t)\} = s^3 \cdot \mathcal{L}\{f(t)\} - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$

Since $f(0) = 1$, $f'(0) = a$, and $f''(0) = a^2$, we have:

$$\mathcal{L}\{f^{(3)}(t)\} = s^3 \cdot \frac{1}{s-a} - s^2 \cdot 1 - s \cdot a - a^2 = \frac{s^3}{s-a} - s^2 - as - a^2$$

Fourth Derivative:

The fourth derivative of $f(t)$ is:

$$f^{(4)}(t) = a^4 e^{at}$$

The Laplace transform of $f^{(4)}(t)$ is:

$$\mathcal{L}\{f^{(4)}(t)\} = s^4 \cdot \mathcal{L}\{f(t)\} - s^3 \cdot f(0) - s^2 \cdot f'(0) - s \cdot f''(0) - f^{(3)}(0)$$

Since $f(0) = 1$, $f'(0) = a$, $f''(0) = a^2$, and $f^{(3)}(0) = a^3$, we have:

$$\mathcal{L}\{f^{(4)}(t)\} = s^4 \cdot \frac{1}{s-a} - s^3 \cdot 1 - s^2 \cdot a - s \cdot a^2 - a^3 = \frac{s^4}{s-a} - s^3 - as^2 - a^2 s - a^3$$