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Mathematics 1

Lecture 11

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الرياضيات الاساسية: المرحلة الاولى

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1 Partial Derivative

Definition 1.1. The **partial derivative** of a function $f(x, y)$ with respect to x, while keeping y constant, is defined as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Remark 1.1. Partial derivatives are commonly represented as:

$$\frac{\partial f}{\partial x}, \quad f_x, \quad D_x f$$

Example 1.1. Consider the function:

$$f(x, y) = x^2 y + 3xy^3$$

The partial derivative with respect to x is:

$$\frac{\partial f}{\partial x} = 2xy + 3y^3$$

The partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = x^2 + 9xy^2$$

Example 1.2. Let $f(x, y) = x^2 - y^2 + xy + 7$. Then find f_x, f_y, f_{xx}, and f_{xy}.

Solution

$$f_x = 2x + y$$

$$f_y = -2y + x$$

$$f_{xy} = 1$$

Problem

1. Let $f(x, y) = e^{-x} \sin y + e^y \cos x + 8$ Then find f_x, f_y.
2. Find f_x and f_y at point $(1, 3/2)$ if $f = \sqrt{4 - x^2 + y^2}$.
3. If $f(x, y) = xe^y - \sin(x/y) + x^3y^2$, then find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.
4. If $V = x^2 + y^2 + z^2 + ln(xz)$, then find V_x, V_y, V_z, V_{xy}, and V_{zz}.
5. If $f = x^y$, then find f_x, f_y.
6. Prove that

$$U_{xy} = U_{yx}$$

7. If:
   (a) $U = x \sin y + y \cos x$
   (b) $U = x \ln y$

1.1 Chain Rule

1- Function of One Variable

If $y = f(x)$, and $x = x(t)$, $y = y(t)$ then:

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial t}$$

2- Function of Two or Three Variables

a. If $z = f(x, y)$, and $x = x(t)$, $y = y(t)$, then:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

b. If $z = f(x, y, w)$, and $x = x(t)$, $y = y(t)$, $w = w(t)$, then:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t}$$

Example 1.3. Let $f(x, y) = e^{xy}$, where $x = r\cos\theta$, $y = r\sin\theta$. Find f_r and f_θ in terms of r and θ.

Solution

First, compute the partial derivatives:

$$\frac{\partial f}{\partial x} = ye^{xy}$$

$$\frac{\partial f}{\partial y} = xe^{xy}$$

$$\frac{\partial x}{\partial r} = \cos\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

Now, using the chain rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= ye^{xy}\cos\theta + xe^{xy}\sin\theta$$

$$= r\sin\theta\cos\theta e^{r^2\cos\theta\sin\theta}$$

$$= r\sin 2\theta \cdot e^{r^2\cos\theta\sin\theta}$$

For f_θ:

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= ye^{xy}(-r\sin\theta) + xe^{xy}(r\cos\theta)$$

$$= -r^2\sin^2\theta e^{r^2\cos\theta\sin\theta} + r^2\cos^2\theta e^{r^2\cos\theta\sin\theta}$$

$$= r^2 e^{r^2\cos\theta\sin\theta}(\cos^2\theta - \sin^2\theta)$$

$$= r^2 e^{r^2\cos\theta\sin\theta}\cos 2\theta$$

1.2 The Total Differential

The total differential of a function $W = f(x, y, z)$ is defined as:

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Or,

$$dw = f_x dx + f_y dy + f_z dz$$

In general, the total differential of a function $W = f(x, y, z, u, ...)$ is defined by:

$$dw = f_x dx + f_y dy + f_z dz + f_u du + \dots$$

where $x, y, z, u, v,$ and w are independent variables.

If $x, y,$ and z are not independent variables but given by:

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

then we have:

$$dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}dt, \quad dz = \frac{dz}{dt}dt$$

Or, in the form:

$$x = x(r, s), \quad y = y(r, s), \quad z = z(r, s)$$

then we have:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial s}ds$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial s}ds$$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial s}ds$$

Thus, equation (1) becomes:

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Substituting equations (2) and (3), we obtain:

$$dw = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}\right)dr + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}\right)ds$$

Example 1.4. Find the total differential of the function:

$$w = x^2 + y^2 + z^2$$

if:

$$x = r\cos s, \quad y = r\sin s, \quad z = r$$

Solution

The total differential is:

$$dw = w_x dx + w_y dy + w_z dz$$

Since:

$$w = x^2 + y^2 + z^2$$

we have:

$$dw = 2xdx + 2ydy + 2zdz$$

Computing differentials:

$$dx = d(r\cos s) = dr\cos s - r\sin s ds$$

$$dy = d(r\sin s) = dr\sin s + r\cos s ds$$

$$dz = dr$$

Now:

$$dw = 2x(\cos s\,dr - r\sin s\,ds) + 2y(\sin s\,dr + r\cos s\,ds) + 2r\,dr$$

$$= 2[x\cos s\,dr - xr\sin s\,ds] + 2[y\sin s\,dr + yr\cos s\,ds] + 2r\,dr$$

$$= 2[r\cos s\cos s\,dr - r^2\cos s\sin s\,ds + r\sin s\sin s\,dr + r^2\sin s\cos s\,ds] + 2r\,dr$$

$$= 2r\left[\cos^2 s + \sin^2 s\right]dr + 2[-r^2\sin s\cos s + r^2\sin s\cos s]ds$$

$$= 2r\,dr$$