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Mathematics 1

Lecture 6

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Contents

1 Chain Rule

The chain rule is a fundamental rule for differentiating composite functions. If a function y is defined as $y = f(u)$, where $u = g(x)$ is a function of x, then the chain rule states:

$$\text{if } y = f(u), \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Example 1.1. Consider $y = \sin(x^2)$. This can be viewed as $y = \sin(u)$ with $u = x^2$. Therefore, we have

$$\frac{dy}{du} = \cos(u) \quad \text{and} \quad \frac{du}{dx} = 2x.$$

Thus, the chain rule can be used to differentiate y with respect to x as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \cos(u) \times (2x)$$

$$= 2x\cos(x^2), \quad \text{since } u = x^2.$$

Example 1.2. Find $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ of $x = 3t + 1$ and $y = t^2$.

Sol.

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t, \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3}.$$

□

Example 1.3. Find the derivative of $y = \cos^2(5x)$.

Sol. Rewrite $y = (\cos(5x))^2$ and let $u = \cos(5x)$, so $y = u^2$.

Then:

$$\frac{dy}{du} = 2u \quad \text{and} \quad \frac{du}{dx} = -5\sin(5x).$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Substitute:

$$\frac{dy}{dx} = 2u \cdot (-5\sin(5x)) = -10\cos(5x)\sin(5x).$$

□

Exercises

Exercise 1: Find $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ for the following:

$$x = 5t - 2, \quad y = t^3.$$

Exercise 2: Find $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ for the following:

$$x = 4t^2 + 1, \quad y = t^4.$$

Exercise 3: Find $\frac{dy}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dx}$ for the following:

$$x = \sin(t), \quad y = \cos(t).$$

Exercise 4: Differentiate $y = \tan^{-1}(2x + 1)$. (Hint: Let $u = 2x + 1$ and use the derivative of $\tan^{-1}(u)$.)

Exercise 5: Differentiate $y = e^{3x^2}$. (Hint: Let $u = 3x^2$.)

2 Implicit Differentiation

Implicit differentiation is used to find derivatives of functions where y is not explicitly expressed as a function of x. It involves differentiating both sides of an equation with respect to x and treating y as a function of x.

Steps for Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x.
2. Apply the chain rule when differentiating y, treating it as a function of x.
3. Solve for $\frac{dy}{dx}$ (the derivative of y).

Example 2.1. Given the equation of a circle:

$$x^2 + y^2 = 1$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

Using the chain rule:

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 2.2. Given the equation:

$$x^2y + xy^2 = 5$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) = \frac{d}{dx}(5)$$

Apply the product rule to each term:

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

Group terms with $\frac{dy}{dx}$:

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

Exercise

Given:

$$\sin(xy) = x + y$$

3 L'Hopital's Rule

L'Hopital's Rule provides a method to evaluate the limits of indeterminate forms, particularly $\frac{0}{0}$,$\frac{\infty}{\infty}$, $0 \times \infty$, $\pm\infty \times \pm\infty$, ∞^∞, 0^0, ∞^0 or $\frac{\infty}{0}$ or $\infty - \infty$. The rule states:

$$\text{If} \quad \lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty,$$

and if $f'(x)$ and $g'(x)$ exist and are continuous near $x = a$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists.

This rule can be applied repeatedly if the limit on the right-hand side still results in an indeterminate form.

Example 3.1 (Applying L'Hopital's Rule)**.** Consider:

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

Sol. First, check the limit:

$$\lim_{x\to 0} \sin(x) = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

so we have the indeterminate form $\frac{0}{0}$.

Apply L'Hopital's Rule by differentiating the numerator and denominator:

$$\frac{d}{dx}[\sin(x)] = \cos(x), \quad \frac{d}{dx}[x] = 1.$$

Now compute the limit:

$$\lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$$

Thus:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

□

Example 3.2. Consider:

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

Sol. First, check the form:

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} x^2 = \infty,$$

so we have the indeterminate form $\frac{\infty}{\infty}$.

Apply L'Hopital's Rule by differentiating the numerator and denominator:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[x^2] = 2x.$$

Now compute the limit:

$$\lim_{x\to\infty} \frac{e^x}{2x}.$$

We still have an $\frac{\infty}{\infty}$ form, so apply L'Hopital's Rule again:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[2x] = 2.$$

Now compute the limit:

$$\lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty.$$

□

Example 3.3. Consider:

$$\lim_{x\to 0} \frac{x - \sin(x)}{x^3}.$$

Sol. First, check the form:

$$\lim_{x\to 0}(x - \sin(x)) = 0 \quad \text{and} \quad \lim_{x\to 0} x^3 = 0,$$

so we have the indeterminate form $\frac{0}{0}$.

Apply L'Hopital's Rule by differentiating the numerator and denominator:

$$\frac{d}{dx}[x - \sin(x)] = 1 - \cos(x), \quad \frac{d}{dx}[x^3] = 3x^2.$$

Now compute the limit:

$$\lim_{x\to 0} \frac{1-\cos(x)}{3x^2}.$$

This still gives $\frac{0}{0}$, so apply L'Hopital's Rule again:

$$\frac{d}{dx}[1-\cos(x)] = \sin(x), \quad \frac{d}{dx}[3x^2] = 6x.$$

$$\lim_{x\to 0} \frac{\sin(x)}{6x} = \frac{0}{0},$$

Apply L'Hopital's Rule again:

$$\frac{d}{dx}[\sin(x)] = \cos(x), \quad \frac{d}{dx}[6x] = 6.$$

Now compute the limit:

$$\lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}.$$

Thus:

$$\lim_{x\to 0} \frac{x-\sin(x)}{x^3} = \frac{1}{6}.$$

□

Exercises on L'Hopital's Rule

Exercise 1: Evaluate the limit using L'Hopital's Rule:

$$\lim_{x\to 0} \frac{\tan(x)}{x}.$$

Exercise 2: Evaluate the limit using L'Hopital's Rule:

$$\lim_{x\to 0} \frac{e^x-1}{x}.$$

Exercise 3: Evaluate the limit using L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{1}{x}.$$

Exercise 4: Evaluate the limit using L'Hopital's Rule:

$$\lim_{x\to 0} \frac{x - \cos(x)}{x^2}.$$

Exercise 5: Evaluate the limit using L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$