Solution of Linear Equations by Cramer's Rule

Cramer's rule is a method for solving linear simultaneous equations. It makes use of determinants and so a knowledge of these is necessary before proceeding.

1. Cramer's Rule - two equations

If we are given a pair of simultaneous equations

$$\begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

then x, and y can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Example
Solve the equations

$$\begin{aligned} 3x + 4y &= -14 \\ -2x - 3y &= 11 \end{aligned}$$

Solution
Using Cramer's rule we can write the solution as the ratio of two determinants.

$$x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2, \qquad y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5$$

The solution of the simultaneous equations is then $x = 2$, $y = -5$.

Example: Use Cramer's rule to solve the simultaneous equations:

$2x+y=7$

$3x-4y=5$

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Calculating $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11 \qquad x = \frac{1}{\Delta}\begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}, \quad y = \frac{1}{\Delta}\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}$$

i.e. $x = \dfrac{(-28-5)}{(-11)}$, $y = \dfrac{(10-21)}{(-11)}$ implying: $x = \dfrac{-33}{-11} = 3$, $y = \dfrac{-11}{-11} = 1$.

You can check by direct substitution that these are the exact solutions to the equations.

2. Cramer's rule - three equations

For the case of three equations in three unknowns: If

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

then x, y and z can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Or,

$$x_1 = \frac{\Delta_{x1}}{\Delta}, \quad x_2 = \frac{\Delta_{x2}}{\Delta}, \quad x_3 = \frac{\Delta_{x3}}{\Delta}$$

Example:

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12. \end{aligned}$$

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Solution:

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix}.$$

$$\begin{aligned} \Delta &= 1 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 1 \times (2-1) + 2 \times (4+3) + 1 \times (-2-3) \\ &= 1 + 14 - 5 = 10 \end{aligned}$$

$$\begin{aligned} 3 \quad \times & \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 5 & 1 \\ 12 & -1 \end{vmatrix} \\ &= 3 \times 1 + 2 \times 22 + 1 \times (-17) \\ &= 30 \end{aligned}$$

Hence $x_1 = \frac{1}{10} \times 30 = 3$

$$\begin{aligned} x_2 &= \frac{1}{10} \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{vmatrix} \\ &= \frac{1}{10} \left\{ 1 \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} \right\} \\ &= \frac{1}{10} \{22 - 3 \times 7 + 9\} = 1 \\ x_3 &= \frac{1}{10} \left\{ 1 \times \begin{vmatrix} 1 & 5 \\ -1 & 12 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \right\} \\ &= \frac{1}{10} \{17 + 2 \times 9 + 3 \times (-5)\} = 2 \end{aligned}$$

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Example : Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Here the determinant of the coefficients is:

$$|A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$
$$= -4(8+3) - 2(6-1) - 9(-9-4)$$
$$= -44 - 10 + 117$$
$$|A| = 63$$

for $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}$$
$$= 2(11) - 2(2) - 9(-47) = 22 - 4 + 423$$

$$\boxed{|A_x| = 441}$$

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}$$
$$= -4(2) - 2(5) - 9(19)$$
$$= -8 - 10 - 171$$

$$\boxed{|A_y| = -189}$$

and

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}$$
$$= -4(47) - 2(19) + 2(-13)$$
$$= -188 - 38 - 26$$

$$\boxed{|A_z| = -252}$$

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Hence $x = \frac{|A_x|}{|A|} = \frac{441}{63} = 7$

$$y = \frac{|A_y|}{|A|} = \frac{-189}{63} = -3$$

$$z = \frac{|A_z|}{|A|} = \frac{-252}{63} = -4$$

<u>Homework</u>: Use Cramer's rule to solve the following sets of simultaneous equations.

$$\begin{aligned} 7x + 3y &= 15 \\ -2x + 5y &= -16 \end{aligned}$$

b)

$$\begin{aligned} x + 2y + 3z &= 17 \\ 3x + 2y + z &= 11 \\ x - 5y + z &= -5 \end{aligned}$$

Answers

a) x = 3, y = −2. b) x = 1, y = 2, z = 4

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