Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

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Lecture Title: [Series]

Series

Infinite Series

Infinite series are sequences of a special kind: those in which the n^{th}-term is the sum of the first n terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots \Rightarrow \boldsymbol{a_1, a_2, a_3, a_4 \ldots \ldots . a_n}$$

If we denote the above sequence as a_n, and the resultant sequence of the series as s_n, then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

Infinite series

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by the rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- The number $s_n = \sum_{k=1}^{n} a_k$ is called the ***n^{th} partial sum*** of the series.
- Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- The series $\sum a_n$ is said to ***converge*** to a number L if and only if $L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

  in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric series

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + + ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r.

$$r = \frac{ar}{r} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \frac{ar^n}{ar^{n-1}}$$

If $|r| < 1$ the series converges to $a/(1-r)$, **[sum of convergence].**

If $|r| \geq 1$, the series diverges unless $a = 0$.

If $a = 0$, the series converges to 0.

Examples

Example

Determine whether each series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$, (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

Solution

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ To be solved as a geometric series, we should satisfy the following form: $a+ar+ar^2+ar^3+\ldots++ar^{n-1}+\ldots$

$$at\ n=0\,,\quad a=\left(\frac{2}{3}\right)^0=1$$

$$at\ n=1,\quad ar=\left(\frac{2}{3}\right)^1=\frac{2}{3}$$

$$at\ n=2,\quad ar^2=\left(\frac{2}{3}\right)^2=\frac{4}{9}$$

$$at\ n=3,\quad ar^3=\left(\frac{2}{3}\right)^3=\frac{8}{27}$$

If $|r|<1$ the series converges to $a/(1-r)$, **[sum of convergence]**.

If $|r|\geq 1$, the series diverges unless $a=0$.

If $a=0$, the series converges to 0.

$$r=\frac{ar}{r}=\frac{ar^2}{ar}=\frac{ar^3}{ar^2}=\frac{ar^n}{ar^{n-1}}\ldots\ldots=\frac{2}{3}$$, so it is a **geometric series**.

$$Since\ \left|r=\frac{2}{3}\right|<1,\qquad the\ geometric\ series\ is\ \textbf{convergent}.$$

$$The\ sum\ of\ convergence\ is\ \frac{a}{1-r}=\frac{1}{\left(1-\frac{2}{3}\right)}=3$$

Examples

In similar way for (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

(b) Since the series is a geometric series with $r=\frac{3}{2}>1$, so the series is divergent.

(c) $\cos\pi/3=1/2$. This is a geometric series with first term $a_1=1$ and the ratio $r=1/2$; so the series converges and its sum is $1/(1-\frac{1}{2})=2$.

(d) $\tan\pi/4=1$. This is a geometric series with $r=1$, so the series diverges.

(e) This is a geometric series with first term $a_1=-5/4$ and ratio $r=-1/4$. So the series converges and its sum is $\frac{-5/4}{1+(1/4)}=-1$.

Test convergence of series with non-negative terms

1) The n^{th} - Term Test

- If $\lim_{n\to\infty} a_n \neq 0$, or if $\lim_{n\to\infty} a_n$ fails to exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0$, then the test fails **(inconclusive test)** ➡ **Choose another test**

Examples

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$,

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$,

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist,

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ can not be tested by the nth-term test for divergence because $\frac{1}{n} \to 0$.

Test convergence of series with non-negative terms

2) The Integral Test

Let the function $y = f(x)$, obtained by introducing the continuous variable x in place of the discrete variable n in the nth-term of the positive series $\sum_{n=1}^{\infty} a_n$, then

$$\int_1^{\infty} f(x)dx = \begin{cases} +\infty & Div. \\ -\infty & Div. \\ -\infty < c < \infty & Conv. \end{cases}$$

Example

Test the convergence of

(a) $\sum_{n=1}^{\infty} \frac{1}{e^n}$, (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

*A special case of integral test is called (**P − series**):*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \Rightarrow \int_1^{\infty} x^{-p} dx$$

If $P \le 1 \rightarrow$ *diverge* (*Harmonic series at* $\mathbf{p = 1}$)

If $P > 1 \rightarrow$ *converge*

Solution

(a) $\int_1^{\infty} e^{-x} dx = -e^{-x}\Big|_1^{\infty} = -(e^{-\infty} - e^{-1}) = \frac{1}{e}$ ***(Conv.)***

(b) $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_2^{\infty} \frac{1/x}{(\ln x)^2} dx = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{-1}{\infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ ***(Conv.)***

Test convergence of series with non-negative terms

3) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- ❖ The series converges if $\rho < 1$,
- ❖ The series diverges if $\rho > 1$,
- ❖ The series may converge or it may diverge if $\rho = 1$. (Test fails) ➡ **(inconclusive test)**

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Test convergence of series with non-negative terms

- The series converges if $\rho < 1$,
- The series diverges if $\rho > 1$,
- The series may converge or it may diverge if $\rho = 1$. (Test fails) ➡ **(inconclusive test)**

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$, (b) $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$, (c) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$, (d) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$, (e) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2! = \cdots$

$(n+1)! = (n+1)\, n! = (n+1)\, n\, (n-1)\,! = \ldots..$

$(n-1)! = (n-1)\,(n-2)! = (n-1)\,(n-2)\,(n-3)\,! = \cdots$

(a) If $a_n = \frac{n!n!}{(2n)!}$, then $a_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{4n+2} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to \frac{1}{4} < 1$$

(Conv.)

Test convergence of series with non-negative terms

$5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2! = \cdots$

$(n+1)! = (n+1)\,n! = (n+1)\,n\,(n-1)\,! = \ldots\ldots$

$(n-1)! = (n-1)\,(n-2)! = (n-1)\,(n-2)\,(n-3)\,! = \cdots$

(b) If $a_n = \dfrac{4^n n! n!}{(2n)!}$, then $a_{n+1} = \dfrac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \times \frac{(2n)!}{4^n n! n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1}$$

$\Rightarrow \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$

$\to 1$ ***(Test fails)***

- ❖ The series converges if $\rho < 1$,
- ❖ The series diverges if $\rho > 1$,
- ❖ The series may converge or it may diverge if $\rho = 1$. (Test fails) ➡ **(inconclusive test)**

(c) If $a_n = \dfrac{2^n + 5}{3^n}$, then $a_{n+1} = \dfrac{2^{n+1} + 5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3} \times \frac{2^{n+1}+5}{2^n+5} \times \frac{2^{-n}}{2^{-n}} = \frac{1}{3} \times \left(\frac{2 + 5 \times 2^{-n}}{1 + 5 \times 2^{-n}} \right)$$

$\Rightarrow \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$

$$\to \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} < 1$$

(Conv.)

Test convergence of series with non-negative terms

- The series converges if $\rho < 1$,
- The series diverges if $\rho > 1$,
- The series may converge or it may diverge if $\rho = 1$. (Test fails) ➡ **(inconclusive test)**

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \Rightarrow \boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n}} \to \infty > 1 \qquad \textit{(Div.)}$$

(e) If $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$ and

Use the property

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n (n+1) n!}{(n+1) n! \, n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \Rightarrow \boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n}} \to e^1 = 2.7 > 1 \qquad \textit{(Div.)}$$

Test convergence of series with non-negative terms

4) The n^{th} Root Test

$$\sqrt[n]{a_n} \to \rho$$

Then

- The series converges if $\rho < 1$.
- The series diverges if $\rho > 1$.
- The test is not conclusive if $\rho = 1$.

Example

Test the convergence of the following series using the n^{th} Root Test.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$, (d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Test convergence of series with non-negative terms

(a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$, (d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

(a_n is the term inside each sum.)

$\sqrt[n]{a_n} \to \rho$

Solution

(a) $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n}$ (ρ) ➡ $\lim_{n\to\infty} \rho \to 0 < 1$ *(Conv.)*

(b) $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\sqrt[n]{n^2}} = \frac{2}{\left(\sqrt[n]{n}\right)^2}$ (ρ) ➡ $\lim_{n\to\infty} \rho \to \frac{2}{1^2} = 2 > 1$ *(Div.)*

(c) $\sqrt[n]{\left(1-\frac{1}{n}\right)^n} = \left(1-\frac{1}{n}\right)$ (ρ) ➡ $\lim_{n\to\infty} \rho \to 1$ *(Test fails)*

Use the property

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ ➡ $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1$

(d) $\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \to \frac{1}{e} = \frac{1}{2.7} < 1$ *(Conv.)*

(e) $\sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \frac{2n}{n+1}$ (ρ) ➡ $\lim_{n\to\infty} \rho \to 2 > 1$ *(Div.)*

H.W.

Find the sum of the following series

$$\sum_{n=1}^{\infty}\frac{7}{4^n}$$

$$\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}$$

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$$

Which of the following series converges and which diverges? Find the sum of the convergent series.

$$\sum_{n=0}^{\infty}e^{-2n}$$

$$\sum_{n=0}^{\infty}\frac{n!}{1000^n}$$

$$\sum_{n=1}^{\infty}\frac{2}{10^n}$$

$$\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$$

$$\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$$

$$\sum_{n=0}^{\infty}\left(\frac{e}{\pi}\right)^n$$

H.W.

Which of the following series converges and which diverges?

$$\sum_{n=1}^{\infty} \frac{-1}{8^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$$

$$\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

$$\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$$

$$\sum_{n=1}^{\infty} n! e^{-n}$$

$$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n}$$

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$$

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$$

$$\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$$

$$\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$$