Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

Subject Name: Mathmatics II 2

2nd Class, Second Semester

Subject Code: [UOMU011042]

Academic Year: 2024-2025

Lecturer: Lecturer: Dr. Ameer Najah

Asst. lec. Eman Yasir

Email: Eman.yasir.hussein @uomus.edu.iq

Lecture No.: 5

Lecture Title: (Series part 2)

Alternating Series

Alternating Series

A series in which the terms are alternately positive and negative, and it has the form:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots - \cdots$$

Example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \frac{(-1)^n 4}{2^n} + \ldots$$

$$1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n+1} n + \ldots$$

The Convergence Test of Alternating Series

The series is **convergent** , if and only if, the following two conditions are satisfied:

1) $a_1 \geq a_2 \geq a_3 \geq a_4 \ldots\ldots$ (***decresing series***)

2) $\lim_{n \to \infty} a_n = 0$

Examples

Example

Test the convergent of the following series:

1) $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n-1}$ 2) $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1}}{n-1}$ 3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Solution: 1) $\sum_{n=1}^{\infty} (-1)^n \boxed{\frac{2n}{4n-1}}$ a_n

1) $a_1 \geq a_2 \geq a_3 \geq a_4 \ldots\ldots$ (*decresing series*)

2) $\lim_{n\to\infty} a_n = 0$

1] $at\ n = 1, \quad a_1 = \frac{2*1}{(4*1)-1} = \frac{2}{3}$

$at\ n = 2, \quad a_2 = \frac{2*2}{(4*2)-1} = \frac{4}{7}$

$at\ n = 3, \quad a_3 = \frac{2*3}{(4*3)-1} = \frac{6}{11}$

$at\ n = 4, \quad a_4 = \frac{2*4}{(4*4)-1} = \frac{8}{15}$

$\frac{2}{3} \geq \frac{4}{7} \geq \frac{6}{11} \geq \frac{8}{15} \ldots\ldots$ (*decresing series*)

2] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n-1} = \frac{\infty}{\infty}$ [**indeterminate form** that should be avoided] by either dividing each term in the nominator and denominator by highest power of n (which is n) or using L'Hopital's Rule. So, $\lim_{n\to\infty} \frac{2n}{4n-1} = \frac{2}{4} = \boxed{\frac{1}{2} > 0}$, *so the condition is not satisfied.*

The series is divergent (Div.) cause condition (2) is not satisfied.

Examples

1) $a_1 \geq a_2 \geq a_3 \geq a_4 \dots\dots$ (*decresing series*)

2) $\lim_{n\to\infty} a_n = 0$

2) $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1}}{n-1}$ ← a_n

1] **Series is decreasing**
Notice that a1 starts at n=2.

2] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n+1}/n}{n-1\,/n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n}{n^2}+\frac{1}{n^2}}}{\frac{n}{n}-\frac{1}{n}} = \frac{0}{1} = 0$ → **Conv.**

3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ ← a_n

1] **Series is decreasing**
Notice that a1 starts at n=1.

2] $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ → **Conv.**

Note:- **The series** $\sum_{n=1}^{\infty} \frac{1}{n}$ **is divergent (Div.) by using either integral test or P-series test.**

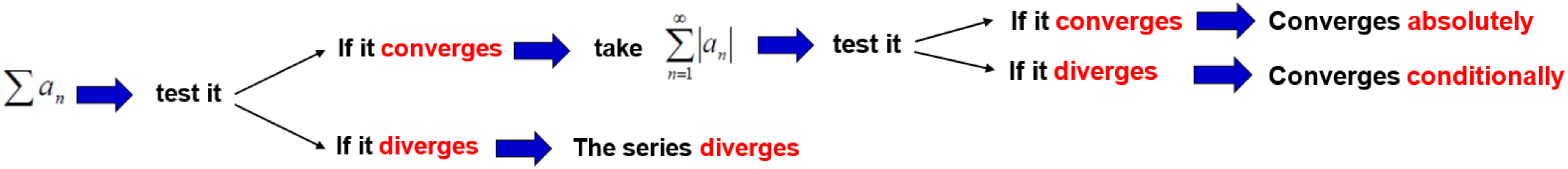
Absolute and Conditional convergence

Absolute Convergence

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Conditional Convergence

A series that converges but does not converge absolutely ***converges conditionally***.



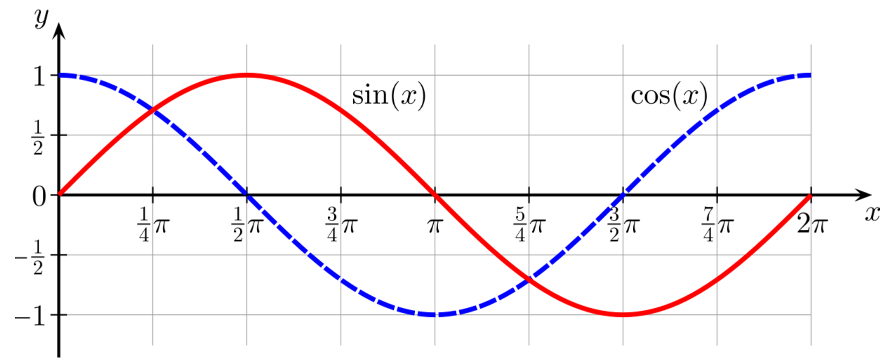
Examples

Example ***Which of the following series converges absolutely, conditionally, and which diverges?***

1) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ 2) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ 3) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$ 4) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

Solution:

1) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ (a_n) → $\cos(n\pi) = \pm 1 \;\; at \;\; n = 0,1,2,3,4,5, \ldots.$



$\sum_{n=1}^{\infty} \frac{\pm 1}{n^{\frac{3}{2}}}$ → it **converges** by either integral test or p-series test ($p = 3/2 > 1$).

Then, take the absolute $\sum_{n=1}^{\infty} \left|\frac{\cos(n\pi)}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \left|\frac{\pm 1}{n^{\frac{3}{2}}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ which **converges** too, so it converges **absolutely**.

Examples

2) $\sum_{n=1}^{\infty} \boxed{\frac{(-100)^n}{n!}}$ (a_n)

$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ → it **converges** by ratio test ($\frac{-100}{n+1} = 0 < 1$).

Then, take the absolute $\sum_{n=1}^{\infty} \left|\frac{(-100)^n}{n!}\right| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n(100)^n}{n!}\right| = \sum_{n=1}^{\infty} \frac{(100)^n}{n!}$ which **converges** too by ratio test, so it converges **absolutely**.

3) $\sum_{n=1}^{\infty} (-1)^{n+1} \boxed{\frac{3+n}{5+n}}$ (a_n)

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$ → it **diverges** (alternating series test) as 2^{nd} condition is not satisfied, so it **diverges** (no need to take absolute for a_n).

4) $\sum_{n=1}^{\infty} (-1)^n \boxed{\frac{1}{\sqrt{n}}}$ (a_n)

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ → it **converges** (alternating series test), two conditions are satisfied.

Then, take the absolute $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which **diverges** by either integral or p-series($p = 1/2 < 1$) tests, so it converges **conditionally**.

Power Series

Power series are defined as infinite series of powers of some variable (x).

In general, infinite series give us precise ways to express many numbers and functions as arithmetic sum with infinitely many numbers. It can extend our knowledge of how to evaluate, differentiate and integrate polynomials for new functions such as differential equations arising in important applications of mathematics to science and engineering.

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}.$$

- A power series about $x = 0$ is a series of the form

u_n

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

- A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$$

in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Convergence of Power Series

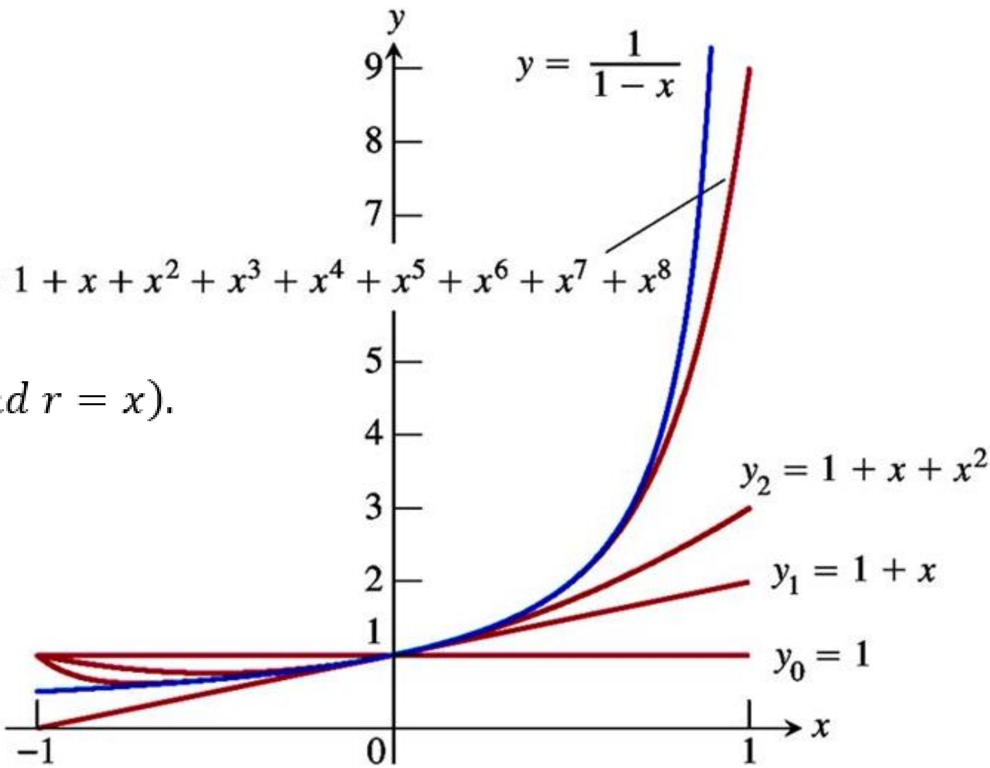
The test of power series is done using the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \rho \begin{cases} <1 & Conv. \\ >1 & Div. \\ =1 & Fails \end{cases}$$

$c_n x^n$

This condition gives us the interval of convergence.

(inconclusive test)



For example

$$The\ series \sum_{n=0}^{\infty} x^n \ \ is\ a\ geomtric\ series\ (where\ a = 1, and\ r = x).$$

$$It\ converges\ to\ the\ sum \left(\frac{a}{1-r}\right) = \frac{1}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n$$

$$For\ |x| < 1 \Rightarrow -1 < x < 1$$

Examples

Find the interval of convergence and test the endpoints.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ (b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ (c) $\sum_{n=0}^{\infty}n!x^n$ (d) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \rho\begin{cases}<1 & Conv. \\ >1 & Div. \\ =1 & Fails\end{cases}$$ **(inconclusive test)**

Solution: u_n

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|(-1)^{n+1-1}\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}.|x|$$

$$= \lim_{n\to\infty}\frac{n}{n+1}.\lim_{n\to\infty}|x| = 1.\lim_{n\to\infty}|x| = 1.|x| = |x| \Rightarrow \rho = |x|$$

To get the interval of convergence, make $\rho < 1$ ⇒ $|x| < 1$ ⇒ $-1 < x < 1$ (endpoints: -1 and 1)

We can, also, get the endpoints by equating ρ *to* 1 $(\rho = 1)$ ⇒ $|x| = 1$ ⇒ $x = \pm 1$

Examples

Now, we test the endpoints (at x =1 and x = -1).

$at\ x = 1,\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ *becomes* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ ➡ It is an alternating series and it converges.

$at\ x = -1,\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ *becomes* $\sum_{n=1}^{\infty} (-1)^{2n-1} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ ➡ It diverges by either integral test or by p-series test (p=1) .

So, the interval of convergence is $\boxed{-1 < x \le 1}$ *and diverges elsewhere.*

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \rho \begin{cases} <1 & Conv. \\ >1 & Div. \\ =1 & Fails \end{cases}$$ (inconclusive test)

$$\rho = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{1}{n+1}.|x|$$

$$= \lim_{n\to\infty} \frac{1}{n+1}.\lim_{n\to\infty} |x| = 0.\lim_{n\to\infty} |x| = 0\ for\ every\ x.$$

Since $\rho < 1$*, it converes absolutely for all x.*

Examples

(c) $\sum_{n=0}^{\infty} n!\,x^n$

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{1}\times\frac{1}{n!\,x^n}\right| = \lim_{n\to\infty} n+1.\,|x|$$

$$= \lim_{n\to\infty} n+1.\ \lim_{n\to\infty}|x| = \infty.\ \lim_{n\to\infty}|x| = \infty \text{ unless } x = 0.$$

Since $\rho > 1$, *it diveres for all values of* x *except* $x = 0$.

(d) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \rho\begin{cases} <1 & \text{Conv.} \\ >1 & \text{Div.} \\ =1 & \text{Fails} \end{cases}$$

(inconclusive test)

$$\rho = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|(-1)^{n+1-1}\frac{x^{2(n+1)-1}}{2(n+1)-1}\times\frac{2n-1}{x^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{2n-1}{2n+1}x^2\right|$$

$$= \lim_{n\to\infty}\frac{2n-1}{2n+1}.\ \lim_{n\to\infty}|x^2| = 1.\,|x^2| = x^2 \Rightarrow \rho = x^2 = |x|$$

To get the interval of convergence, make $\rho < 1$ ➡ $|x| < 1$ ➡ $-1 < x < 1$

endpoints

$at\ x = 1,\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ *becomes* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}$

$at\ x = -1,\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ *becomes* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{-1}{2n-1}$

both are alternating series and they converge.

So, the interval of convergence is $-1 \le x \le 1$ *and diverges elsewhere.*

Taylor and Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the ***Taylor Series*** generated by f at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The ***Maclaurin Series*** generated by f is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

which is a Taylor series generated by f at $x = 0$.

Examples

Find Maclaurin series and Taylor series generated by f (x) at x = a for the following function $f(x) = e^{-2x}$, $a = 3$

Solution:

Maclaurin

$$f(x) = e^{-2x} \rightarrow f(0) = 1$$
$$f^{-}(x) = -2e^{-2x} \rightarrow f^{-}(0) = -2$$
$$f^{=}(x) = 4e^{-2x} \rightarrow f^{=}(0) = 4$$
$$f^{\equiv}(x) = -8e^{-2x} \rightarrow f^{\equiv}(0) = -8$$
$$f^{4}(x) = 16e^{-2x} \rightarrow f^{4}(0) = 16$$

Taylor

$$f(a = 3) = e^{-6} = 0.00248$$
$$f^{-}(a = 3) = -2e^{-6} = -0.0049$$
$$f^{=}(a = 3) = 4e^{-6} = 0.0099$$
$$f^{\equiv}(a = 3) = -8e^{-6} = -0.0198$$
$$f^{4}(a = 3) = 16e^{-6} = 0.0396$$

The **Maclaurin** series is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$

$$f(x) = e^{-2x} = 1 - 2x + 4\frac{x^2}{2!} - 8\frac{x^3}{3!} + 16\frac{x^4}{4!} - \cdots + \cdots$$

The **Taylor** series is $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$

$$f(x) = e^{-2x} = e^{-6} - 2e^{-6}(x-3) + 4e^{-6}\frac{(x-3)^2}{2!} - 8e^{-6}\frac{(x-3)^3}{3!} + 16e^{-6}\frac{(x-3)^4}{4!} - \cdots + \cdots$$

Examples

Find the Taylor series for the following function $f(x) = \ln(x)$ at $x = 1$.

Solution:

$$f(x) = \ln(x), \qquad f(1) = 0,$$

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1,$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2,$$

$$f^{(4)}(x) = \frac{-6}{x^4}, \qquad f^{(4)}(1) = -6,$$

The **Taylor** series is $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$

$$\ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \ldots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n + \ldots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + \frac{(-1)^{n+1}}{n}(x-1)^n + \ldots$$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

H.W.

Which of the following series converges and which diverges?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^{n}$$

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln\left(n^{2}\right)}$$

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$$

Which of the following series converges absolutely, conditionally, and which diverges?

$$\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^{n}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}(n+1)^{n}}{(2n)^{n}}$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{(2n)!}{2^{n}n!n}$$

H.W.

Find the interval of convergence and test the endpoints.

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{2^n n}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$$

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$$

Find Maclaurin series and Taylor series generated by f(x) at x = a for the following functions:

$$f(x) = \frac{1}{x^2},\ a = 1$$

$f(x) = 7\cos(-x)$, a =2