Root Locus Technique.

Root Locus

It is a graphical method, in which movement of poles in the s-plane is sketched when a particular parameter of system is varied from zero to infinity. For root locus method, gain (K) is assumed to be a parameter which is to be varied from zero to infinity.

ROOT-LOCUS PLOTS

Angle and magnitude conditions. Consider the system shown in Figure 6–1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad (6\text{–}1)$$

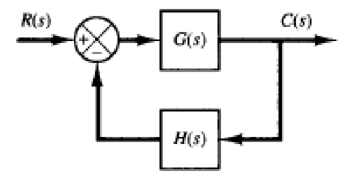


Figure 6–1 Control system.

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (6–1) equal to zero. That is,

$$1 + G(s)H(s) = 0 \qquad \text{or} \qquad G(s)H(s) = -1 \qquad (6\text{–}2)$$

Angle condition:

$$\angle G(s)H(s) = \pm 180^\circ(2k + 1) \qquad (k = 0, 1, 2, \ldots) \qquad (6\text{–}3)$$

Magnitude condition:

$$|G(s)H(s)| = 1 \qquad (6\text{–}4)$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

A plot of the points in the complex plane satisfying the angle condition alone is the root locus.

Email (abdulrahim.thiab.humod@uomus.edu.iq)

F

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$

where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where ϕ_1, θ_1, θ_2, θ_3, and θ_4 are measured counterclockwise as shown in Figures 6–2(a) and (b). The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

where A_1, A_2, A_3, A_4, and B_1 are the magnitudes of the complex quantities $s + p_1, s + p_2$, $s + p_3, s + p_4$, and $s + z_1$, respectively, as shown in Figure 6–2(a).

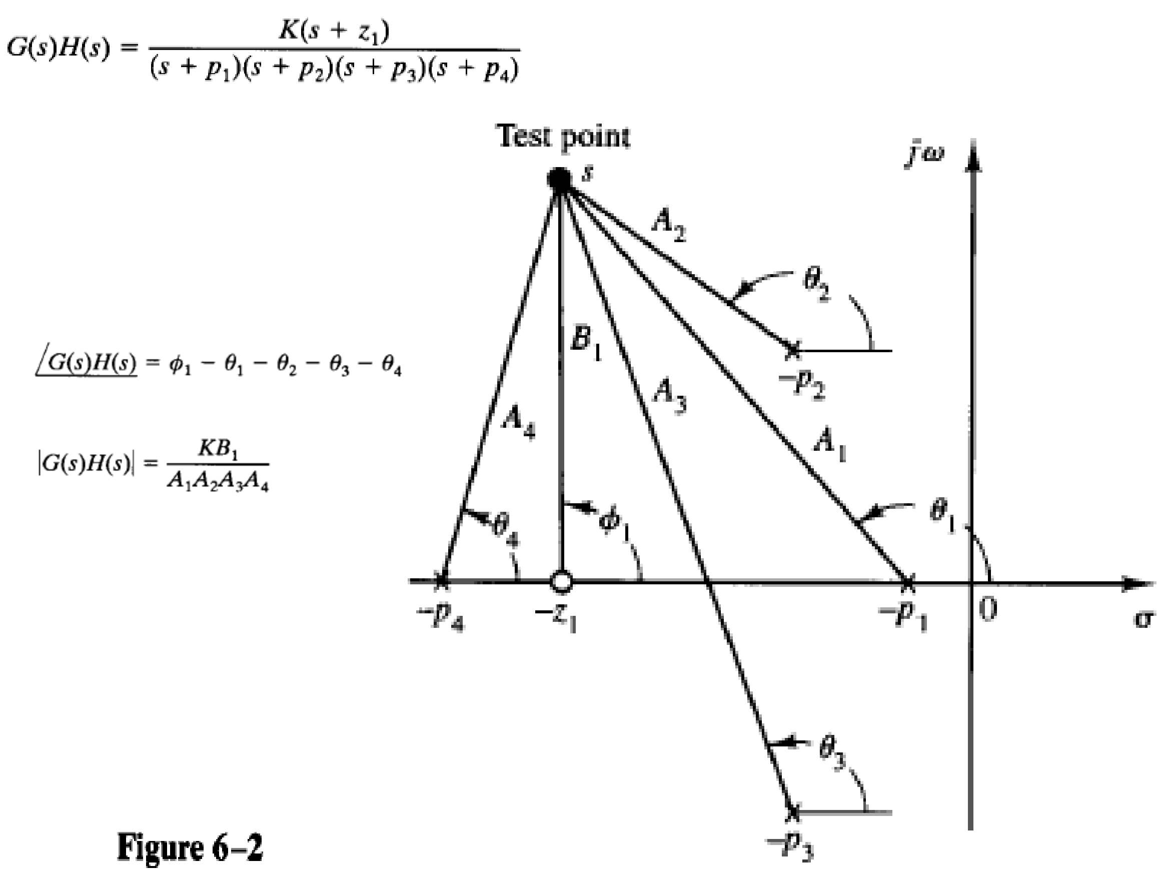


Figure 6–2

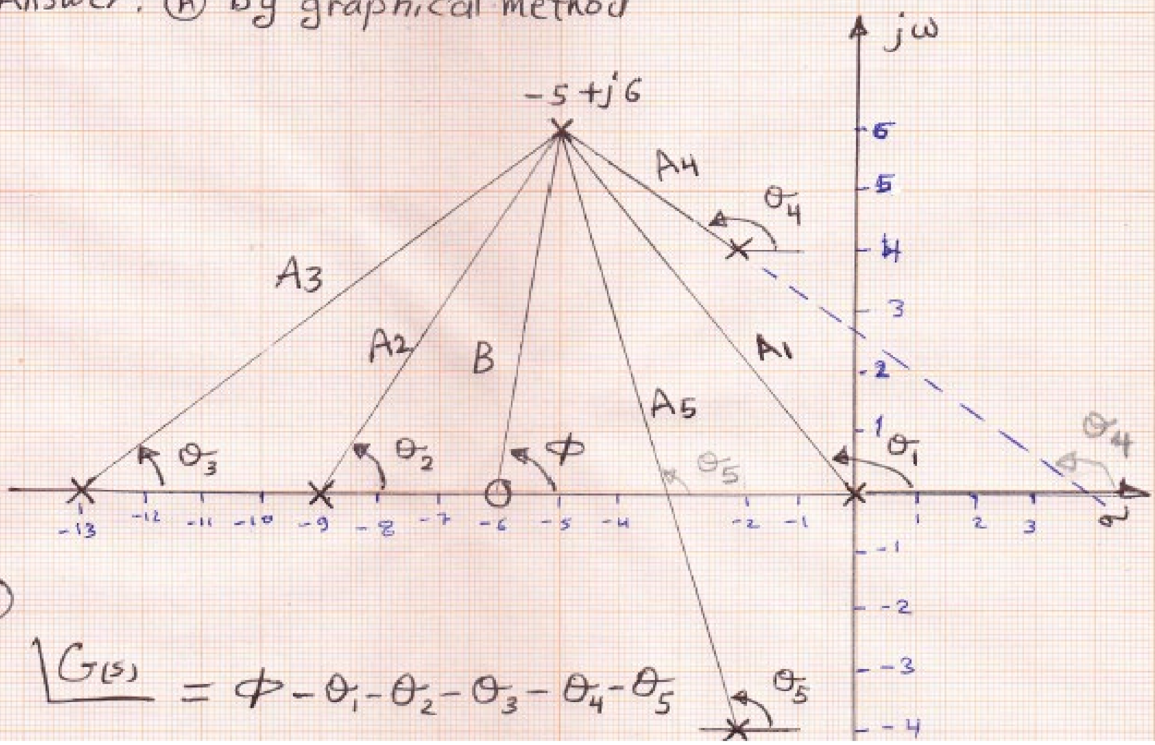
Example : Consider the following open loop Transfer function

$$G(s) = \frac{4(s+6)}{s(s+9)(s+13)(s+2-j4)(s+2+j4)}$$

For the test point $s = -5 + j6$

1- Determine the angle condition
2- Determine the magnitude condition
3- Is the test point part of root locus ?

Answer: (A) by graphical method



(1)

$$\angle G(s) = \phi - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5$$

$$\angle G(s) = 81^\circ - 129^\circ - 56^\circ - 36^\circ - 146^\circ - 107^\circ = -393^\circ$$

(2)

$$|G(s)| = \frac{4B}{A_1 A_2 A_3 A_4 A_5} = \frac{4 \times 6.1}{7.7 \times 7.2 \times 10 \times 3.5 \times 10.4}$$

$$|G(s)| = 0.0012$$

(3) $-393 \neq \pm 180(2k+1)$

The point is not part of root locus

Email (abdulrahim.thiab.humod@uomus.edu.iq)

Rule 7: Centroid of the asymptotes—The point of intersection of the asymptotes with the real axis is called the centroid σ_A which is calculated as

$$\sigma_A = \frac{\Sigma \text{ Real parts of poles} - \Sigma \text{ Real parts of zeros}}{P - z}$$

Let us consider the example of Rule 6 where $G(s) = K/s(s+2)$

$$\sigma_A = \frac{[0-2]-[0]}{2-0} = -\frac{2}{2} = -1$$

Rule 8:Breakaway points—The root locus breakaway from the real axis where a number of roots are available, normally, where two roots exist.

Consider example 9.4, $G(s)H(s) = \frac{K}{s(s+2)}$

$s = 0$ and $s = -2$

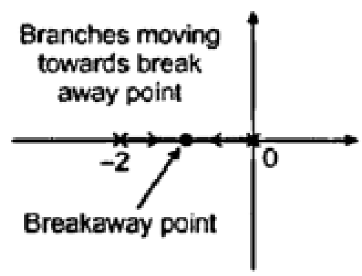


Fig 9.3

Determination of breakaway point :

Steps to determine the co-ordinates of breakaway points are,

Step 1 : Construct the characteristic equation $1 + G(s)H(s) = 0$ of the system.

Step 2 : From this equation, separate the terms involving 'K' and terms involving 's'. Write the value of K in terms of s.

$$K = f(s)$$

Step 3 : Differentiate above equation w.r.t. 's', equate it to zero.

$$\frac{dK}{ds} = 0$$

Step 4 : Roots of the equation $\frac{dK}{ds} = 0$ gives us the breakaway points.

Key Point: *If value of K is positive that breakaway point is valid for the root locus. The breakaway points for which values of K are negative, are invalid for direct root locus but are valid for inverse root locus.*

Example 9.1 : *For* $G(s)H(s) = \dfrac{K}{s(s+1)(s+4)}$, *determine the co-ordinates of valid breakaway points.*

Solution : Characteristic equation $1 + G(s)H(s) = 0$

Step 1 : $1 + \dfrac{K}{s(s+1)(s+4)} = 0$ i.e. $s^3 + 5s^2 + 4s + K = 0$

Step 2 : $K = -s^3 - 5s^2 - 4s$

Step 3 : $\dfrac{dK}{ds} = -3s^2 - 10s - 4 = 0$

Step 4 : $3s^2 + 10s + 4 = 0$

$\therefore$ Breakaway points $= \dfrac{-10 \pm \sqrt{100 - 4\times4\times3}}{2\times3} = -0.46, -2.86$

Substituting in expression for K

For $s = -0.46$, $K = +0.8793$

For $s = -2.86$, $K = -6.064$

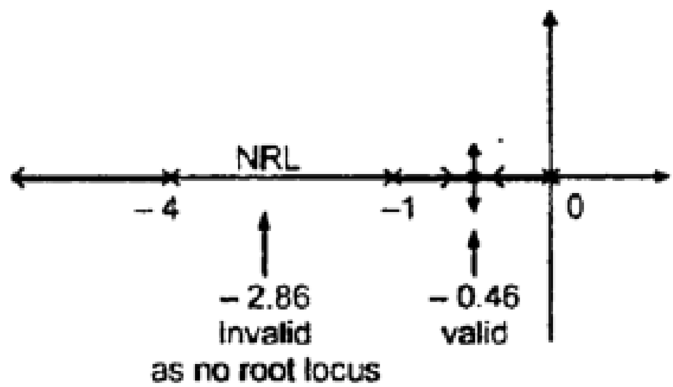


Fig. 9.4

Rule 9: Intersection of root locus with imaginary axis. This can be determined by following procedure.

Step 1 :	Consider characteristic equation $1 + G(s)H(s) = 0$ as obtained in Rule 8
Step 2 :	Construct Routh's array in terms of "K".
Step 3 :	Determine $K_{marginal}$ i.e. value of K which creates one of the rows of Routh's array as row of zeros, except the row of s^0.
Step 4 :	Construct auxiliary equation $A(s) = 0$ by using coefficients of a row which is just above the row of zeros.
Step 5 :	Roots of auxiliary equation $A(s) = 0$ for $K = K_{mar}$ are nothing but the intersection points of the root locus with imaginary axis.

Consider example 9.1 :

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

Characteristic equation is given by,

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+4)} = 0$$

i.e. $s^3 + 5s^2 + 4s + K = 0$

Routh's array,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

K_{mar} = 20 that makes row corresponding to s^1 as row of zeros.

$$\therefore \quad A(s) = 5s^2 + K = 0$$

$$K = K_{mar} = 20$$

$$5s^2 + 20 = 0$$

$$s^2 = -4 \quad \therefore \quad s = \pm j2$$

> **Key Point:** *If K_{mar} is positive, root locus intersects with imaginary axis. But if K_{mar} is negative root locus does not intersect with imaginary axis and lies totally in left half of s-plane.*

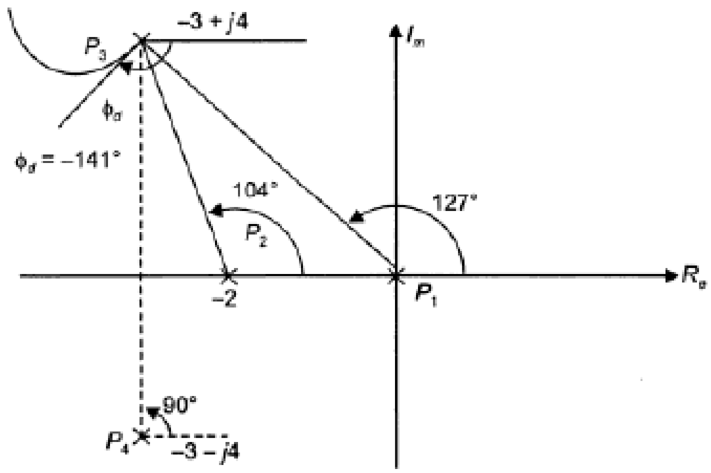
Rule 10: Angle of departure of the root locus—The angle of departure of the locus from a complex pole is calculated as

ϕ_d = 180° – sum of angles made by vectors drawn from the other poles to this pole + sum of angles made by vectors drawn from the zeros to this pole.

Let us consider an example. Let

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+25)}$$

$$\phi_d = 180° - (127° + 104° + 90°) + 0 = 180° - 321° = -141°$$



Example 9.2 : *For* $G(s)H(s) = \dfrac{K(s+2)}{s(s+4)(s^2+2s+2)}$, *calculate angles of departures at complex conjugate poles.*

Solution : P = 4 , Z = 1

Poles are at s = 0 , – 4 , – 1 ± j

Zero at s = –2.

Draw Pole-Zero plot.

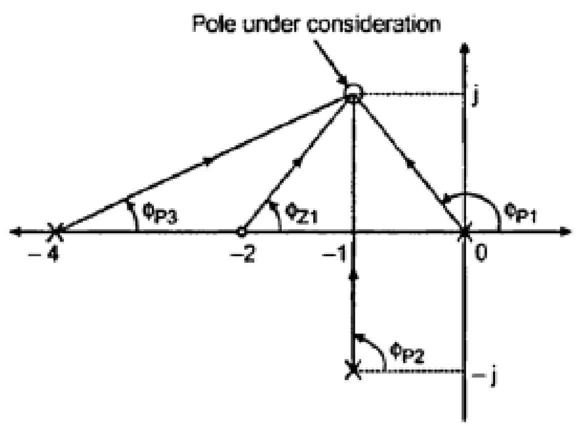


Fig. 9.5

Then , $\sum \phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3}$ while

$$\sum \phi_Z = \phi_{Z1}$$

From geometry of the Fig. 9.17 we can calculate,

$$\phi_{P1} = 135°, \quad \phi_{P2} = 90°, \quad \phi_{P3} = 18.43°$$

$$\therefore \quad \sum \phi_P = 135° + 90° + 18.43° = 243.43°$$

$$\sum \phi_Z = \phi_{Z1} = 45°$$

$$\therefore \quad \phi = \sum \phi_P - \sum \phi_Z = 243.43° - 45° = 198.43°$$

$$\phi_d = 180° - \phi = 180° - 198.43° = -18.43°$$

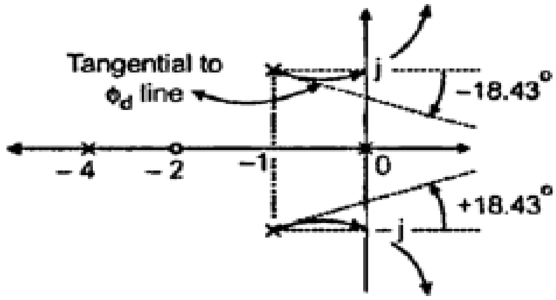


Fig. 9.6

Rule 11: ***Angle of arrival at a complex zero :***

Angle of arrival at a complex zero can be calculated by the same method, which is denoted as ϕ_a. The only change to calculate the angle of arrival is,

$$\phi_a = 180^\circ + \phi$$

where $$\phi = \Sigma \phi_P - \Sigma \phi_Z$$

Obtaining G(s)H(s) from Characteristic Equation

i) Collect the terms of s without K together.

ii) Collect the terms of K together.

iii) Divide the entire equation by polynomial containing the terms of s without K.

This gives the form of equation as 1 + G(s)H(s) = 0.

For example, if the characteristic equation is given as,

$$s^3 + 7s^2 + 12s + Ks + 10K = 0$$

Then rewrite the equation as,

$$(s^3 + 7s^2 + 12s) + K(s + 10) = 0$$

Then divide entire equation by polynomial in s without K i.e.

$$1 + \frac{K(s+10)}{s^3 + 7s^2 + 12s} = 0$$

$$\text{i.e. } 1 + \frac{K(s+10)}{s(s+3)(s+4)} = 0$$

Comparing this with 1 + G(s)H(s) = 0 we get,

$$G(s)H(s) = \frac{K(s+10)}{s(s+3)(s+4)}$$

From this root locus can be obtained.

General Steps to Solve the Problem on Root Locus

Step 1 : Get the general information about number of open loop poles, zeros, number of branches etc. from G(s)H(s) .

Step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions.

Step 3 : Calculate angles of asymptotes.

Step 4 : Determine the centroid. Sketch a separate sketch for step 3 and step 4.

Step 5 : Calculate the breakaway and breakin points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as breakaway points.

Step 6 : Calculate the intersection points of root locus with the imaginary axis.

Step 7 : Calculate the angles of departures or arrivals if applicable.

Step 8 : Combine steps 1 to 7 and draw the final sketch of the root locus.

Step 9 : Predict the stability and performance of the given system by using the root locus.

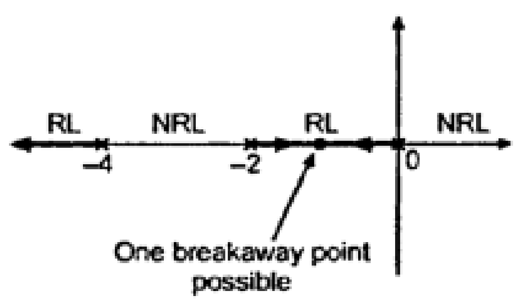
➡ **Example** 9.3 : *For a unity feedback system,* $G(s) = \dfrac{K}{s\,(s+4)\,(s+2)}$. *Sketch the rough nature of the root locus showing all details on it. Comment on the stability of the system.*

(M.U. : June-92)

Solution : Step 1 : General information from $G(s)H(s) = \dfrac{K}{s\,(s+2)\,(s+4)}$

P = 3, Z = 0, number of branches N = P = 3.

Step 2 : Pole-Zero plot and sections of real axis.



Email (abdulrahim.thiab.humod@uomus.edu.iq)

Step 3 : Angles of asymptotes.

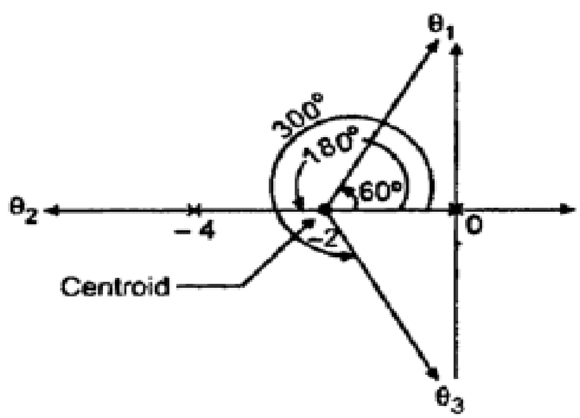
3 branches are approaching to ∞ , 3 asymptotes are required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \qquad q = 0, 1, 2$$

$$\therefore \ \theta_1 = \frac{180^\circ}{3} = 60^\circ, \ \theta_2 = \frac{(2+1)180^\circ}{3} = 180^\circ, \ \theta_3 = \frac{(2\times2+1)180^\circ}{3} = 300^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0-2-4}{3} = -2$$



Step 5 : To find breakaway point (Refer Rule No. 6). Characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$\therefore \quad s^3 + 6s^2 + 8s + K = 0$$

$$\therefore \quad K = -s^3 - 6s^2 - 8s \quad \text{... (1)}$$

$$\frac{dK}{ds} = -3s^2 - 12s - 8 = 0$$

i.e. $$3s^2 + 12s + 8 = 0$$

Roots i.e. breakaway points $= \dfrac{-12 \pm \sqrt{144 - 4\times3\times8}}{2\times3} = -0.845, -3.15$

For $s = -3.15$, $K = -3.079$ (Substituting in equation for K)

For $s = -0.845$ $K = +3.079$

As K is positive $s = -0.845$ is vaild breakaway point.

Email (abdulrahim.thiab.humod@uomus.edu.iq)

Step 6 : Intersection point with imaginary axis.

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

Routh's array

s^3	1	8
s^2	6	K
s^1	$\frac{48-K}{6}$	0
s^0	K	

$K_{marginal}$ = 48 which makes row of s^1 as row of zeros.

$$A(s) = 6s^2 + K = 0$$

$$K_{mar} = 48$$

$$\therefore \; 6s^2 + 48 = 0$$

$$s^2 = -8$$

$$\therefore \quad s = \pm j\sqrt{8} = \pm j2.828$$

Intersection of root locus with imaginary axis is at $\pm j\,2.828$ and corresponding value of $K_{mar} = 48$.

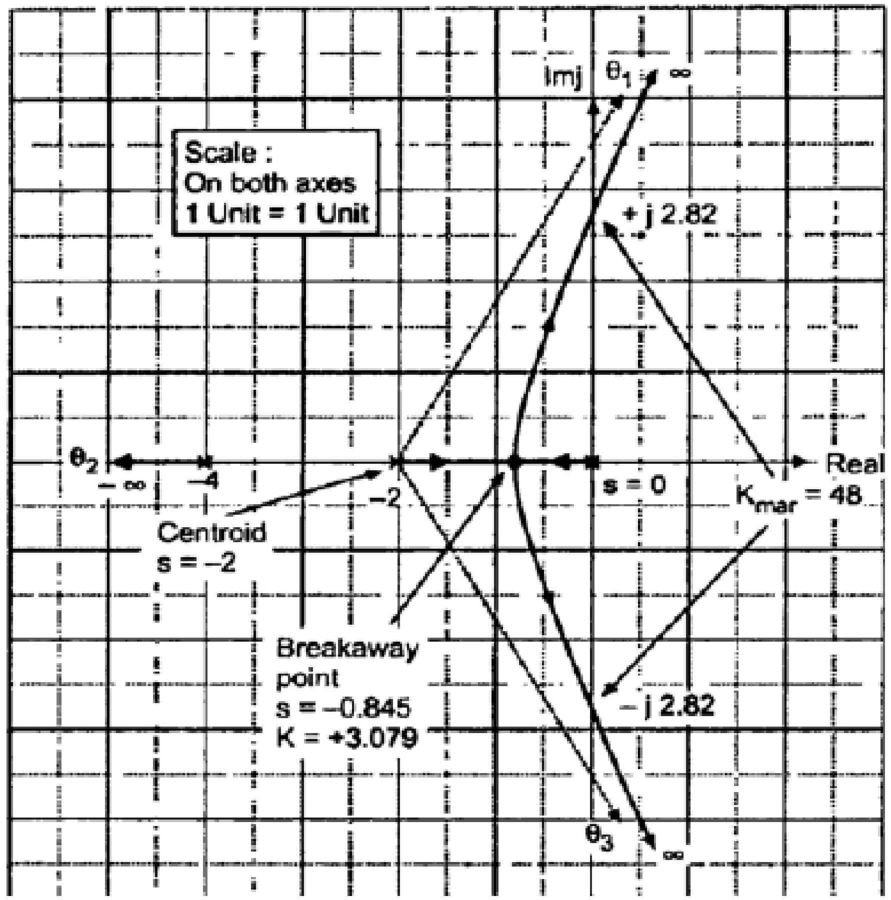
Step 7 : As there are no complex conjugate poles or zeros, no angles of departures or arrivals are required to be calculated.

Step 8 : The complete root locus is as shown below.

Step 9 : Prediction about stability :

For $0 < K < 48$, all the roots are in left half of s–plane hence system is absolutely stable. For $K_{mar} = +48$, a pair of dominant roots on imaginary axis with remaining root in left half. So system is marginally stable oscillating at 2.82 rad/sec. For $48 < K < \infty$, dominant roots are located in right half of s-plane hence system is unstable.

Email (abdulrahim.thiab.humod@uomus.edu.iq)



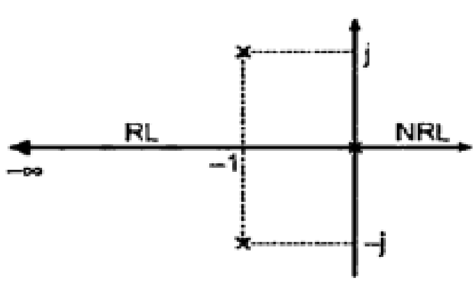
Example 9.4 : *Sketch the root locus for the system having* $G(s)H(s) = \frac{K}{s(s^2+2s+2)}$.

Solution : Step 1 : P = 3, Z = 0, N = P = 3

P – Z = 3 branches approaching to ∞ . Starting points open loop poles,

s = 0, s = – 1 + j , s = –1 – j.

Step 2 : Pole-Zero plot and sections of real axis.



Email (abdulrahim.thiab.humod@uomus.edu.iq)

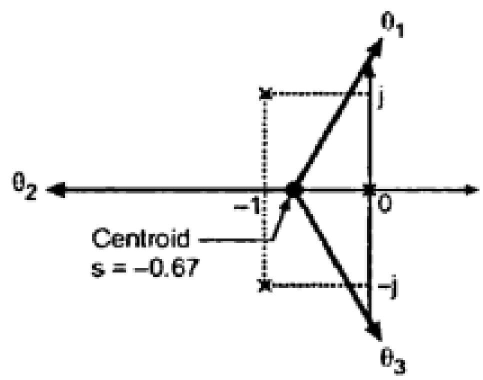
Step 3 : Angles of asymptotes : 3 branches approaching to ∞, 3 asymptotes required

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \quad q = 0, 1, 2.$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ, \ \theta_2 = \frac{(2+1)\,180^\circ}{3} = 180^\circ, \theta_3 = \frac{(2\times 2+1)\,180^\circ}{3} = 300^\circ$$

Step 4 : Centroid : $\sigma = \frac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P-Z} = \frac{0-1-1-0}{3}$

$$= -\frac{2}{3} = -0.67$$



Step 5 : **no breakaway point existing for this system.**

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^3 + 2s^2 + 2s + K = 0$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	0
s^0	K	

$K_{mar} = +4$ makes row of $s^1 = 0$

$A(s) = 2s^2 + K = 0$

At $K_{mar} = 4$

$2s^2 + 4 = 0$

$s^2 = -2 \ \therefore \ s = \pm j\,1.414$

Email (abdulrahim.thiab.humod@uomus.edu.iq)

Step 7 : Angle of departure : As branch is departing at $-1+j$ let us calculate angle of departure, at $-1+j$.

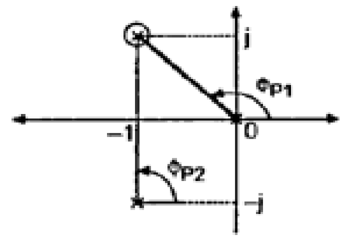
$\phi_{P1} = 135°$, $\phi_{P2} = 90°$

$\Sigma\phi_P = \phi_{P1} + \phi_{P2} = 225°$, $\Sigma\phi_Z = 0$

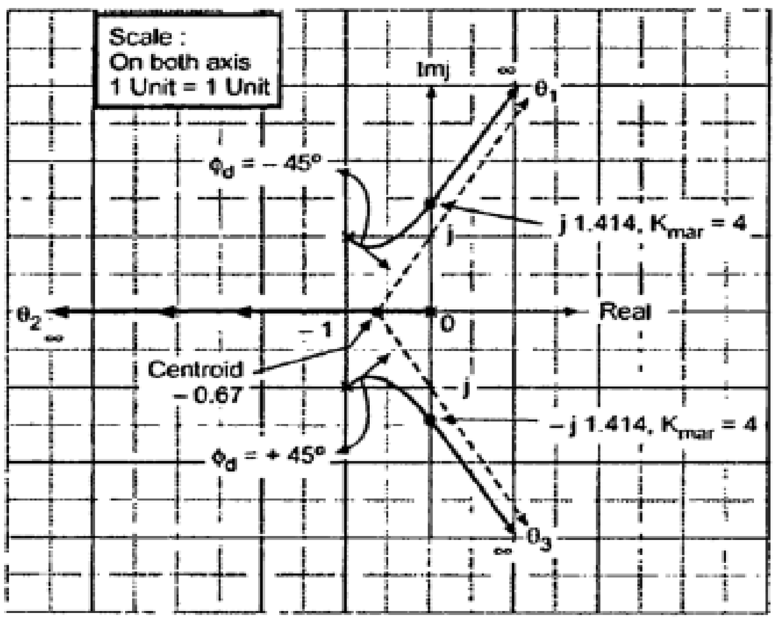
$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 225°$

$\therefore \quad \phi_d = 180° - \phi = 180° - 225° = -45°$

At $-1-j$, $\quad \phi_d = +45°$



Step 8 : Complete Root Locus is :



Step 9 : Comment on stability :

For $0 < K < 4$ all roots are in left half of s-plane. System is absolutely stable.

At $K = +4$, dominant roots are on imaginary axis, system is marginally stable, oscillating with 1.414 rad/sec.

At $K > 4$, dominant roots are in right half of s-plane and hence system becomes unstable in nature.

Example 9.5 : *Sketch the complete root locus for the system having*

$$G(s)H(s) = \frac{K(s+5)}{(s^2+4s+20)}.$$

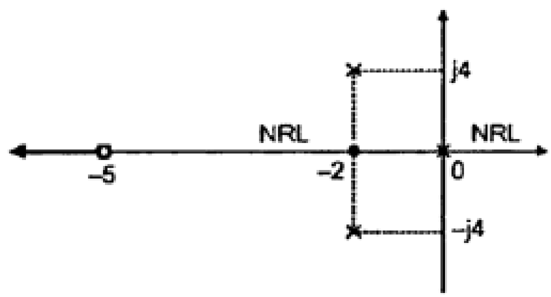
(M.U. : Dec.-2007)

Solution : Step 1 : Number of poles P = 2, Z = 1, N = P

One branch has to terminate at finite zero $s = -5$ while $P - Z = 1$ branch has to terminate at ∞. Starting points of branches are,

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j\,4$$

Step 2 : Pole - Zero plot and sections of real axis are as in following figure.



Step 3 : Angles of asymptotes

One branch approaches to ∞ so one asymptote is required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \quad , q = 0$$

$$\therefore \qquad \theta_1 = 180^\circ$$

Branch approaches to ∞ along + 180° i.e. negative real axis.

Step 4 : Centroid

As there is only one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway point

Characteristic equation : 1 + G(s)H(s) = 0

$$1 + \frac{K(s+5)}{(s^2+4s+20)} = 0$$

$$\therefore \qquad s^2 + 4s + 20 + Ks + 5K = 0$$

$$\therefore \qquad s^2 + 4s + 20 + K(s+5) = 0$$

$$\therefore \qquad K = \frac{-s^2 - 4s - 20}{(s+5)}$$

Now $$\frac{dK}{ds} = \frac{vu' - uv'}{v^2} = 0$$

$$= (s+5)(-2s-4) - (-s^2-4s-20)(1) = 0$$

$$= -2s^2 - 14s - 20 + s^2 + 4s + 20 = 0$$

Email (abdulrahim.thiab.humod@uomus.edu.iq)

i.e. $-s^2 - 10s = 0$

$\therefore \quad -s(s+10) = 0$

$s = 0$ and $s = -10$ are breakaway points. But $s = 0$ cannot be breakaway point as for $s = 0$, $K = -4$.

For $s = -10$, $\quad K = \dfrac{-100+40-20}{-10+5} = +16$

Hence $s = -10$ is valid breakaway point.

Step 6 : Intersection with imaginary axis.

Characteristic equation

$$s^2 + 4s + 20 + Ks + 5K = 0$$

$$s^2 + s(K+4) + (20+5K) = 0$$

Routh's array

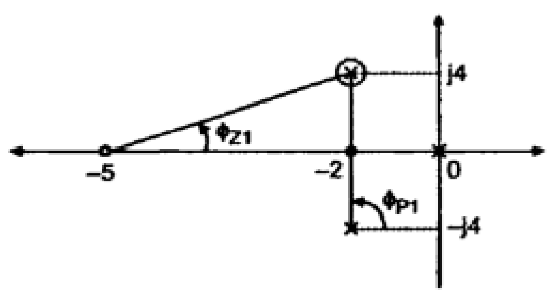
s^2	1	20 + 5 K
s^1	K + 4	0
s^0	20 + 5 K	

$K_{mar} = \quad -4$ makes s^1 row as row of zeros.

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure

Consider $-2 + j4$ join remaining pole and zero to it.



$$\phi_{P1} = 90°, \quad \phi_{Z1} = \tan^{-1}\frac{4}{3} = 53.13°$$

$$\Sigma\phi_{P} = 90°, \quad \Sigma\phi_Z = 53.13°$$

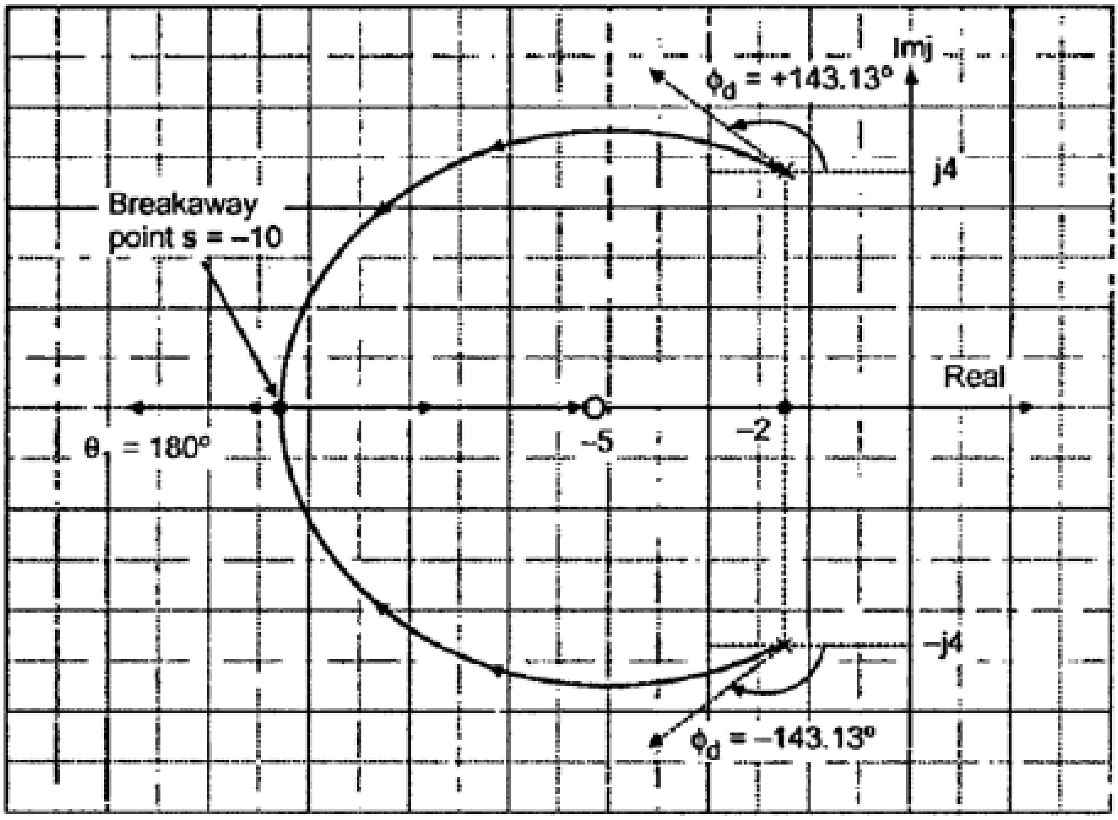
$$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 36.86°$$

$$\therefore \quad \phi_d = 180° - \phi = +143.13° \quad \text{at } -2+j4 \text{ pole}$$

$$\phi_d = -143.13° \quad \text{at } -2-j4 \text{ pole.}$$

Email (abdulrahim.thiab.humod@uomus.edu.iq)

Step 8 : Complete Root Locus is using following figure.



Step 9 : Prediction of stability

For all ranges of K i.e. $0 < K < \infty$, both the roots are always in left half of s–plane. So system is inherently stable.

The Root Locus Method