Sequences & Series

Sequences & Series

Sequences :

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

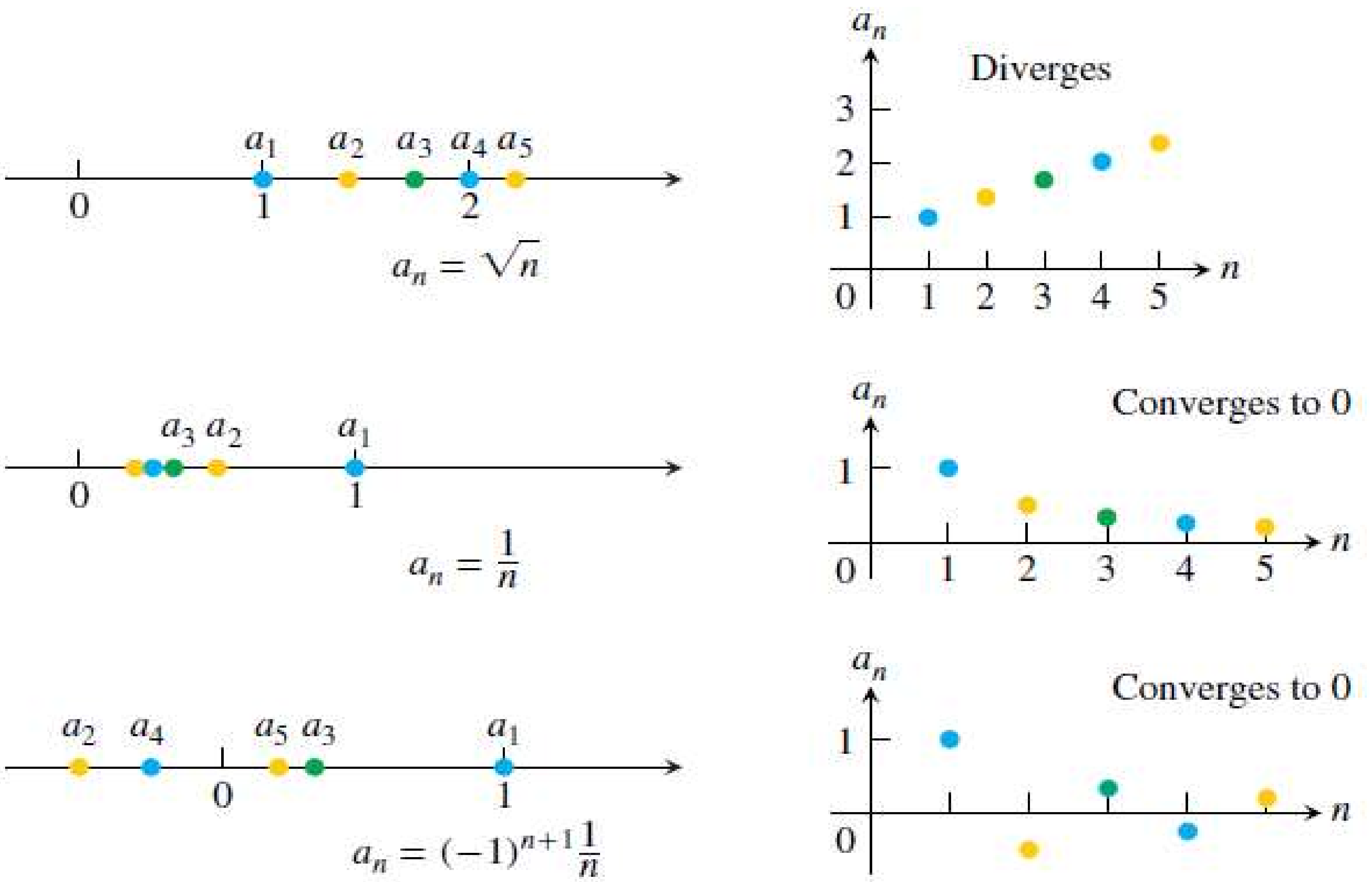
EX:

$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$

$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$

$$\{d_n\} = \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.$$



a_1 a_2 a_3 a_4 a_5
0 1 2
$a_n = \sqrt{n}$
a_n
Diverges
3
2
1
0 1 2 3 4 5 n
a_3 a_2 a_1
0 1
$a_n = \frac{1}{n}$
a_n
Converges to 0
1
0 1 2 3 4 5 n
a_2 a_4 a_5 a_3 a_1
0 1
$a_n = (-1)^{n+1}\frac{1}{n}$
a_n
Converges to 0
1
0
n

DEFINITIONS Converges, Diverges, Limit

The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 11.2).

EXAMPLE 1 Applying the Definition

Show that

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim_{n\to\infty} k = k$ (any constant k)

Solution

(a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty}(1/n) = 0$.

(b) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k. ■

EXAMPLE 2 A Divergent Sequence

Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

Solution Suppose the sequence converges to some number L. By choosing $\epsilon = 1/2$ in the definition of the limit, all terms a_n of the sequence with index n larger than some N must lie within $\epsilon = 1/2$ of L. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\epsilon = 1/2$ of L. It follows that $|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$. But the number L cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit L exists and so the sequence diverges.

Theorem:

$$\lim_{n\to\infty}\frac{f(x)}{g(x)}=\lim_{n\to\infty}\frac{f\prime(x)}{g\prime(x)}$$ (L'Hopital's Rule)

يستخدم في حالة التعويض وينتج $\frac{0}{\infty}$., $\frac{\infty}{\infty}$, $\frac{0}{0}$

EX: Use L'Hopital's Rule to find

$$\lim_{n\to\infty}\frac{2^n}{5n}.$$

By l'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty}\frac{2^n}{5n}=\lim_{n\to\infty}\frac{2^n\cdot\ln 2}{5}$$
$$=\infty.$$

Applying L'Hôpital's Rule to Determine Convergence

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) && \infty \cdot 0 \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} && \frac{0}{0} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} && \text{l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.
\end{aligned}$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2.

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Ex: Check the convergence of the following sequences :

1- $a_n = \sqrt{\frac{n+1}{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} = \sqrt{\lim_{n\to\infty} (1+\frac{1}{n})} = \sqrt{1+\frac{1}{\infty}} = \sqrt{1+0}$$

$$= 1\ (conv.)$$

2-$a_n = (1-\frac{3}{x})^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1-\frac{3}{x})^n = \lim_{n\to\infty} (1+\frac{-3}{x})^n = e^{-3}$$ conv. (from 5)

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also
$\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$ Multiply s_n by r.

$$s_n - rs_n = a - ar^n$$ Subtract rs_n from s_n. Most of the terms on the right cancel.

$$s_n(1 - r) = a(1 - r^n)$$ Factor.

$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1).$$ We can solve for s_n if $r \neq 1$.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 11.1) and $s_n \to a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

Ex: $\sum_{n=0}^{\infty} (\frac{1}{2})^n$ is a G.S.

a=1 ,$r = \frac{1}{2} < 1$ $\therefore$ converge to $\frac{1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

EX: $\sum_{n=0}^{\infty} 3^n$ divergence series because r=3>1

Express the repeating decimal 5.232323 . . . as the ratio of two integers.

Solution We look for a pattern in the sequence of partial sums that might lead to a formula for s_k. The key observation is the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Removing parentheses and canceling adjacent terms of opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}.$$

We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Tests of convergences :

nth term test for divergence :

for series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergence

but $\lim_{n\to\infty} a_n = 0$ then this doesn't mean that $\sum a_n$ is converge .

EX:

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

The integral test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1},$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We have convergence for $p > 1$ but divergence for every other value of p.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by the Integral Test. The function $f(x) = 1/(x^2 + 1)$ is positive, continuous, and decreasing for $x \geq 1$, and

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2 + 1}\, dx &= \lim_{b\to\infty} \Big[\arctan x\Big]_1^b \\ &= \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

EX:

$\sum_{n=2}^{\infty} \frac{1}{nlnn}$,,,, f(x)= $\frac{1}{xlnx}$

$\int_2^{\infty} f(x)dx = \lim_{n\to\infty} (\int_2^{\infty} \frac{1}{xlnx} dx) = \lim_{n\to\infty} (\ln(lnx)]_2^n =$

$\lim_{n\to\infty} (lnlnn - lnln2) = \infty$

$\therefore \sum_{n=2}^{\infty} \frac{1}{nlnn}$ is diverges

The ratio test :

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

(a) For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges because $\rho = 2/3$ is less than 1. This does *not* mean that 2/3 is the sum of the series. In fact,

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The root test :

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho .$$

Then

(a) the series *converges* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

$\sum_{n=1}^{\infty}(1-\frac{3}{n})^{7n^2}$

$\lim_{n\to\infty} \sqrt[n]{(1-\frac{3}{n})^{7n^2}} = \lim_{n\to\infty}(1-\frac{3}{n})^{7n} =$

$(e^{-3})^7 = e^{-21} < 1$

Alternating Series :

A series of form $\sum_{n=0}^{\infty}(-1)^n a_n$ is called **Alternating Series** i.e.

$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 - \cdots \ldots.$

or $\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty} (\cos n\pi)\, a_n$

The Alternating Series Test :

The series $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergence if :

1. $a_n > 0 \quad (a_n \ is \ positive)$
2. $a_n \geq a_{n+1}$ for all n $\geq N$,for some integer N
3. $\lim_{n\to\infty} a_n = 0$

Ex:

1 $- \sum_{n=0}^{\infty}(-1)^n \frac{1}{n}$ is converge since $\lim_{n\to\infty} \frac{1}{n} = 0$

2. $\sum_{n=0}^{\infty} \frac{(\cos n\pi)}{1+n^2} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{1+n^2}$ is converge since $\lim_{n\to\infty} \frac{1}{1+n^2} = 0$

Note:

1. If $\sum|(-1)^n a_n|$ is converge then $\sum(-1)^n a_n$ is converge

If $\sum(-1)^n a_n$ is diverge then $\sum|(-1)^n a_n|$ is also diverge

The Absolutely & Conditional Convergence:

1. If $\sum(-1)^n a_n$ is convergence .this series is called **Absolutely Convergent** if $\sum|(-1)^n a_n|$ is converge.
2. If $\sum(-1)^n a_n$ is convergence and $\sum|(-1)^n a_n|$ is divergence then $\sum(-1)^n a_n$ is called **Conditional Convergent**

1. $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is conv. but $\sum\left|\frac{(-1)^n}{n}\right| = \sum_{n=0}^{\infty}\frac{1}{n}$ is diverge

$\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is **Conditionally Convergent**

$2 - \sum_{n=0}^{\infty}(-1)^n\frac{n}{n+1}$ is divergence because $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$

Power Series :

This has the form $\sum_{n=1}^{\infty} a_n(x-h)^n \;= a_1(x-h) + a_2(x-h)^2 + \; a_3(x-h)^3 \ldots \ldots ..$

To study these series we find the interval of x for absolute convergence by using the ratio test .

EX: Find the interval of absolute convergence of :

1. $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

Using ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| < 1 \quad = \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| < 1$$

=0<1 for every value of x

$\therefore$ interval of conv. is $\infty > x > -\infty$

$\sum_{n=0}^{\infty} 3^n \frac{(x+5)^n}{4^n}$

Using ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\lim_{n\to\infty} \left| 3^{n+1} \frac{(x+5)^{n+1}}{4^{n+1}} . \ . \frac{4^n}{3^n (x+5)^n} \right| < 1 \quad = \lim_{n\to\infty} \left|\frac{3}{4}(x+5)\right| < 1$

= $-1 < \frac{3}{4}(x+5) < 1$

$\frac{-4}{3} <$ x+5$< \frac{4}{3}$

$-\frac{19}{3} < x < \frac{-11}{3}$ radius of conv. R= 4/3

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x - 2) + \frac{f''(2)}{2!}(x - 2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x - 2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x - 2)}{2^2} + \frac{(x - 2)^2}{2^3} - \cdots + (-1)^n \frac{(x - 2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x - 2)/2$. It converges absolutely for $|x - 2| < 2$ and its sum is

$$\frac{1/2}{1 + (x - 2)/2} = \frac{1}{2 + (x - 2)} = \frac{1}{x}.$$