Al- Mustaqbal University

College of Science

Medical Physics Department

First Stage

Mechanics

Lecture two: Oscillations

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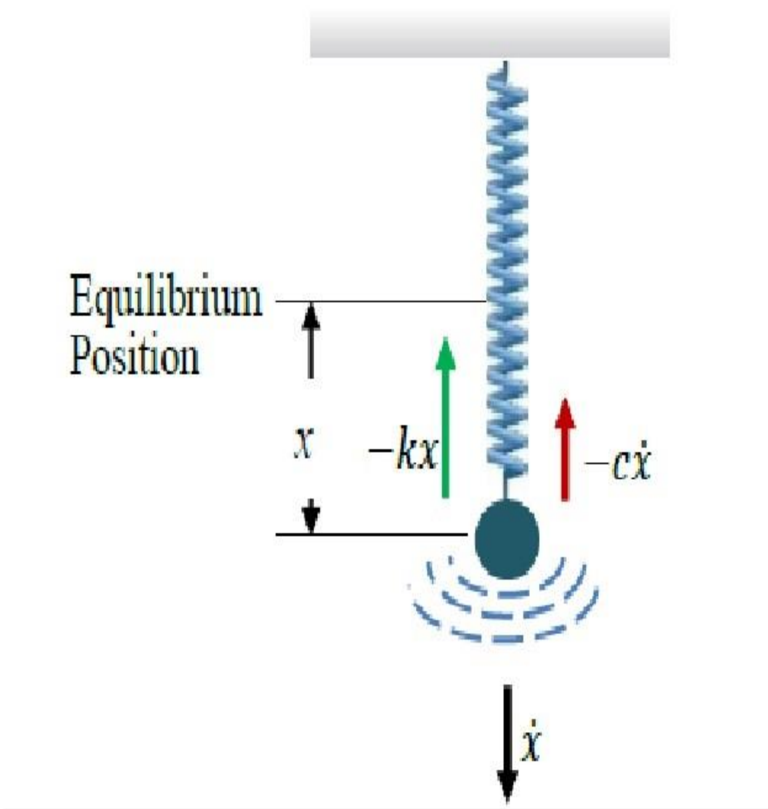
Damped Harmonic Motion

The foregoing analysis of the harmonic oscillator is *idealized* in that we didn't take into account *frictional forces*. These are always present in any mechanical system. Consider an object is supported by a spring of stiffness k and there was a ***viscous retarding force*** varying linearly with the speed such as ***air resistance***.

التحليل السابق لمتذبذب توافقي مثالي لأنه لم ياخذ بنظر الاعتبار قوة الاحتكاك والتي تكون دائما موجودة في أي نظام ميكانيكي. لنفترض أن جسمًا ما معلقاً بنابض معامل مرونته k وكانت هناك قوة معيقة لزجة متغيرة خطياً مع السرعة مثل مقاومة الهواء.



$F = -kx$ …… (1) $(\boldsymbol{restoring\ force})$

$F = -c\,\dot{x}$ …… (2) $(\boldsymbol{retarding\ force})$

Equation of motion then:

$$\therefore -kx - c\dot{x} = m\ddot{x} \quad \text{……. (3)}$$

$m\ddot{x} + c\dot{x} + kx = 0$ … (4) ***Differential Eq. of Motion for Damped Harmonic Oscillator***

Use ***trail method*** to solve Eq. (4)

$$x = A\,e^{qt}$$

$$\therefore\ m\frac{d^2}{dt^2}(A\,e^{qt}) + c\frac{d}{dt}(A\,e^{qt}) + k\,A\,e^{qt} = 0$$

$$m\,q^2\,A\,e^{qt} + cq\,A\,e^{qt} + k\,A\,e^{qt} = 0] \ \div Ae^{qt}$$

$$mq^2 + cq + k = 0 \quad \text{…… (5)} \quad \textbf{\textit{Auxiliary Equation}}$$

$$q = \frac{-c \mp \sqrt{c^2 - 4mk}}{2m} \quad \text{…… (6)}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac_1}}{2a}$$

- $c^2 > 4mk$ **(*Over Damping*)**

Here q will be real and negative and the motion will be nonoscillatory $q_1 \neq q_2$ and (x) decaying to zero exponentially with time.

المقدار $\boldsymbol{c^2 - 4mk}$ يحدد نوع التذبذب

حالة $\boldsymbol{c^2 > 4mk}$ تمثل حالة فوق التضاؤل وعندها q تمتلك قيمتين حقيقيتين سالبتين مختلفتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة x أسياً الى الصفرمع الزمن

$$q = -\begin{cases}\gamma_1 \\ \gamma_2\end{cases} \quad \Rightarrow x = \begin{cases}A_1 e^{-\gamma_1 t} \\ A_2 e^{-\gamma_2 t}\end{cases}$$

The general solution for displacement is:

$$x = A_1\,e^{-\gamma_1 t} + A_2\,e^{-\gamma_2 t} \quad \text{…… (7)}$$

- $c^2 = 4mk$ **(*Critical Damping*)**

Here q will be real also, and negative and the motion will be nonoscillatory. (x) decaying to zero exponentially with time but in shorter time.

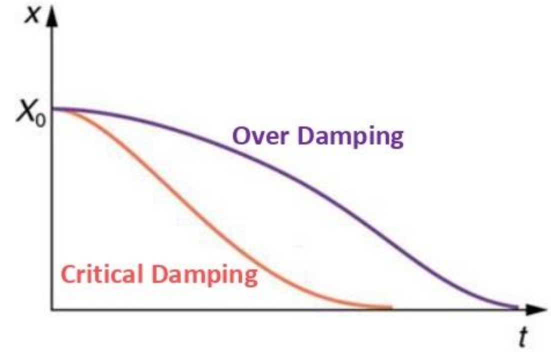
$$q_1 = q_2 = -\frac{c}{2m}$$

$$q = -\gamma \quad \Rightarrow x = \begin{cases}A_1 e^{-\gamma t} \\ A_2 t e^{-\gamma t}\end{cases}$$

The general solution for displacement is:

$$x = A_1\, e^{-\gamma t} + A_2 t\, e^{-\gamma t} \;\ldots\ldots (8)$$

$$x = e^{-\gamma t}(A_1 + tA_2) \;\ldots\ldots (9)$$



حالة $\boldsymbol{c^2 = 4mk}$ تمثل حالة التضاؤل الحرج وعندها q تمتلك قيمتين حقيقيتين سالبتين متساويتين لذلك تكون الحركة غير تذبذبية وتهبط فيها قيمة الازاحة x أسياً الى الصفرمع الزمن

- $\boldsymbol{c^2 < 4mk}$ ***(Under Damping)***

Here q will be complex; the real part of its value gives an oscillatory motion.

$$q = \frac{-c \mp \sqrt{c^2 - 4mk}}{2m}$$

$$q = \frac{-c \mp \sqrt{\left(c^2 \cdot \frac{4m^2}{4m^2} - 4mk \cdot \frac{4m^2}{4m^2}\right)}}{2m}$$

$$q = \frac{-c \mp \sqrt{4m^2\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m} = \frac{-c \mp 2m\sqrt{\left(\frac{c^2}{4m^2} - \frac{k}{m}\right)}}{2m}$$

$$\gamma = \frac{c}{2m} \rightarrow \gamma^2 = \frac{c^2}{4m^2}, \quad w_0 = \sqrt{\frac{k}{m}}$$

$$q = \frac{-c \mp 2m\sqrt{\gamma^2 - w_0{}^2}}{2m}$$

$$q = -\frac{c}{2m} \mp \sqrt{\gamma^2 - w_0{}^2}$$

$$q_{1,2} = -\frac{c}{2m} + i\sqrt{w_0{}^2 - \gamma^2} = -\gamma \mp i\, w_1 \quad \textit{Complex Conjugates Roots}$$

where $w_1 = \sqrt{w_0{}^2 - \gamma^2}$ is ***Natural Frequency***

$$q_1 = -\gamma + i\, w_1$$

$$q_2 = -\gamma - i\, w_1$$

The Displacement then:

$$\therefore x = A_+\, e^{(-\gamma + i\, w_1)t} + A_-\, e^{(-\gamma - i\, w_1)t} \;\ldots\ldots (10)$$

$$\therefore x = e^{-\gamma t}\left(A_+\, e^{i\, w_1 t} + A_-\, e^{-i w_1 t}\right)$$

$$e^{i\, u} = \cos u\, t + i \sin u \quad \textit{Euler's Formula}$$

$$x = e^{-\gamma t}\left[(i\,A_+ - i\,A_-)\sin w\,t + (A_+ + A_-)\cos w_1 t\right]$$

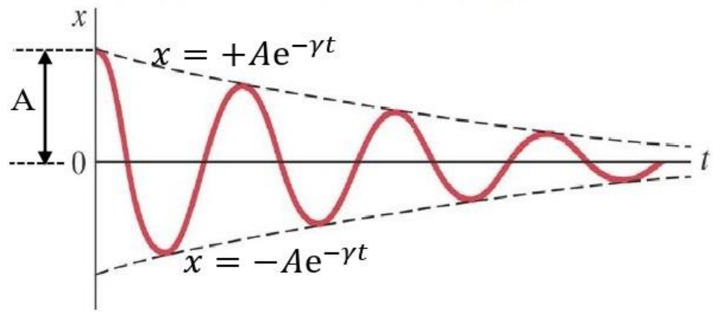
$$x = e^{-\gamma t}\,(\,a\sin w_1 t + b\cos w_1 t\,)$$

where $a = i\,(A_+ - A_-)$, $\quad b = A_+ + A_-$

or $x = A\,\mathrm{e}^{-\gamma t}\cos(w_1 t + \theta_0)$ ……. (11)

where $\theta_0 = \tan^{-1}\frac{b}{a}$

$$A = (a^2 + b^2)^{1/2}$$



Equation (11) shows that the ***two curves*** are given by $x = +A\mathrm{e}^{-\gamma t}$ and $x = +A\mathrm{e}^{-\gamma t}$ form an ***envelope*** of the curve of motion because the cosine factor takes on values between +1 and -1, including +1 and -1, at which points the curve of motion touches the envelope. Accordingly, the points of contact are separated by a time interval of ***one-half period***.

في حالة $c^2 < 4mk$ والتي تمثل حالة دون التضاؤل عندها نحصل على قيمتين غير حقيقيتين (خيالية) لـq والحركة هنا تكون تذبذبية والسعة تتضائل اسياً مع الزمن.

تظهر المعادلة (11) وجود منحنيتين هما $x = +A\mathrm{e}^{-\gamma t}$ و $x = +A\mathrm{e}^{-\gamma t}$ يشكلان غلافاً لمنحنى الحركة لأن عامل الجيب تمام يأخذ القيم بين 1+ و 1- ، بضمنها 1+ و 1- ، والتي يمس فيها منحنى الحركة ، الغلاف. وفقًا لذلك ، لذلك تنفصل نقاط التماس بفترة زمنية مقدارها نصف مدة الذبذبة.

Energy Consideration for Damped Harmonic Oscillator

The total energy of the damped harmonic oscillator is given by the sum of the kinetic and potential energies

الطاقة الكلية للمتذبذب التوافقي المضمحل في اية لحظة هي مجموع للطاقات الحركية والكامنة

$$E_t = E_k + E_p$$

$E_t = \frac{1}{2}m\,\dot{x}^2 + \frac{1}{2}k\,x^2 \dots\dots (1)$

To find ***time rate*** of change of E_t, we have to differentiate E_t with respect to t:

$\frac{dE_t}{dt} = \frac{1}{2}2m\dot{x}\frac{d\dot{x}}{dt} + \frac{1}{2}2kx\frac{dx}{dt}$

$= m\ddot{x}\dot{x} + k\dot{x}x$

$\frac{dE_t}{dt} = (m\ddot{x} + kx)\dot{x} \dots\dots (2)$

We have the Eq. of motion for the damped harmonic oscillator

$m\ddot{x} + c\dot{x} + kx = 0$

$m\ddot{x} + kx = -c\dot{x} \dots\dots (3)$

Sub. Eq. (3) in Eq. (2)

$\therefore \frac{dE_t}{dt} = -c\dot{x}^2 \dots\dots (4)$

This equation represents the rate at which the energy E_t dissipated as frictional heat by virtue of the viscous resistance to the motion.

هذه المعادلة تمثل معدل تبدد الطاقة الكلية الى حرارة بسبب الاحتكاك وهي مقدار سالب دائماً

Forced Harmonic Motion (Resonance)

In this section, we study the motion of a damped harmonic oscillator that is subjected to a periodic driving force by an external agent.

Consider a damped harmonic oscillator motion affected by an external force (F_{ext}) that varying as a ***cosine*** wave with time t, the ***angular frequency*** w and ***amplitude*** (F_0) such that

هنا ندرس حركة المتذبذب التوافقي المضمحل المدفوع بقوة خارجية توافقية. اي قوة تتغير بدالة جيبية مع الزمن

$F_{ext} = F_0 \cos(wt + \theta) \dots\dots (1)$

$F_{ext} = F_0\, e^{i(wt+\theta)} \dots\dots (2)$

There are three forces attached on the body:

هناك ثلاث قوى مؤثرة في الجسم:

1. Elastic restoring Force $= -kx$
2. The viscous damping force $= -c\dot{x}$
3. External force $= F_{ext}$

- قوة معيدة مرنة $(-kx)$
- قوة لزوجة مضمحلة $(-c\dot{x})$
- قوة خارجية (F_{ext})

عليه تكون القوة الكلية المؤثرة على الجسم مجموع لهذه القوى الثلاث

So, total force is:

$$\therefore -kx - c\dot{x} + F_{ext} = m\ddot{x} \;\dots\dots (3)$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = F_0\, e^{i(wt+\theta)} \;\dots\dots (4)$$

Eq.(4) represent differential damped harmonic oscillator motion affected by an external force (F_{ext}). Suggested solution of this equation as:

معادلة (4) تمثل معادلة الحركة لمتذبذب توافقي مضمحل تحت تأثير قوة خارجية (F_{ext})

$$x = A\, e^{i(wt+\theta')} \;\dots\dots (5)$$

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt} A\, e^{i(wt+\theta')}$$

$$\dot{x} = i\, A\, w\, e^{i(wt+\theta')} = iwx \quad \dots\dots (6)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} A\, e^{i(wt+\theta')}$$

$$\ddot{x} = i^2\, A\, w^2\, e^{i(wt+\theta')} = i^2 w^2 x = -w^2 x \;\dots\dots (7)$$

Sub. Eqns.(5) ,(6) and (7) in Eq. (4).

$$-mAw^2 e^{i(wt+\theta')} + cAwi\, e^{i(wt+\theta')} + kAe^{i(wt+\theta')} = F_0\, e^{i(wt+\theta)}] * e^{-i(wt+\theta')}$$

$$\therefore\; -mA\, w^2 + iwcA + kA = F_0 e^{i(wt+\theta)} . e^{-i\,(wt+\,\theta')}$$

$$-mAw^2 + iwcA + kA = F_0\, [e^{iwt}\, e^{i\theta}\, e^{-iwt}\, e^{-i\theta'}\,]$$

$$-mAw^2 + iwcA + kA = F_0 e^{i(\theta-\theta')}$$

$$-mAw^2 + iwcA + kA = F_0[\cos(\theta-\theta') + i\sin(\theta-\theta')]$$

where $\varphi = (\theta - \theta') \equiv Phase\ difference\ (\boldsymbol{Phase\ angle})$

Separation between real and imaginary terms, we get:

$$-mA\, w^2 + kA = F_0 \cos(\theta-\theta') = F_0 \cos\varphi$$

$$iwcA = iF_0 \sin(\theta-\theta) = i\, F_0 \sin\varphi$$

$$A(k - mw^2) = F_0 \cos\varphi \;\dots\dots (8)$$

$$cwA = F_0 \sin\varphi \;\dots\dots (9)$$

Dividing Eq. (9) on Eq. (8)

$$\frac{cw}{k-mw^2} = \frac{F_0 \sin\varphi}{F_0 \cos\varphi} = \tan\varphi \;\dots\dots (10)$$

$$\therefore \tan\varphi = \frac{\frac{c}{m}w}{\frac{k}{m}-w^2}$$

$$\therefore \tan\varphi = \frac{2\gamma\, w}{{w_0}^2-w^2} \;\ldots\ldots (11)$$

$$\frac{c}{2m} = \gamma \qquad \therefore\ \frac{c}{m} = 2\gamma$$

$$w_0 = \sqrt{\frac{k}{m}}$$

Squaring and adding Eqns. (8) and (9)

$$A^2(k-mw^2)^2 + c^2w^2A^2 = F_0^2(\cos^2\varphi + \sin^2\varphi) = F_0^2 \;\ldots\ldots (12)$$

$$A^2\,[(k-mw^2)^2 + c^2w^2] = F_0^2$$

$$A^2 = \frac{F_0^2}{(k-mw^2)^2 + c^2w^2}$$

$$\therefore A = \frac{F_0}{\sqrt{(k-mw^2)^2+c^2w^2}} \;\ldots\ldots (13)$$

by dividing the numerator and denominator on m

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{k}{m}-\frac{mw^2}{m}\right)^2+\frac{c^2w^2}{m}}}$$

So, in term of γ and w_0

$$A = \frac{F_0/m}{\sqrt{(w_0^2-w^2)^2+4\gamma^2 w^2}} \;\ldots\ldots (14)$$ ***Steady State Oscillation Amplitude***

Eq. (14) represent the amplitude (A) as a function of the driving frequency (w). The maximum value of amplitude valid only at ($w = w_0$) (***Resonance Frequency***). To find this frequency equal ***differential amplitude equation by zero.***

معادلة (14) تمثل السعة (A) كدالة للتردد الدافع (w). القيمة القصوى للسعة تتحقق فقط عند ($w = w_0$) (تردد الرنين).

$$\frac{dA}{dw} = \frac{d}{dw}\left[\frac{F_0/m}{\sqrt{(w_0^2-w^2)^2+4\gamma^2w^2}}\right]$$

$$= \frac{F_0}{m}\frac{d}{dw}\,[(w_0^2-w^2)^2+4\gamma^2w^2]^{-\frac{1}{2}}$$

$$= \frac{F_0}{m}\frac{d}{dw}\,[w_0^4+w^4-2w_0^2\,w^2+4\gamma^2w^2]^{-\frac{1}{2}}$$

$$\frac{dA}{dw} = \frac{F_0}{m}\left(\frac{-1}{2}\right)[\,w_0^4 + w^4 - 2w_0^2\,w^2 + 4\gamma^2 w^2]^{-\frac{3}{2}} \cdot [0 + 4w^3 - 4w_0^2 w + 8\gamma^2 w]$$

$$= \frac{F_0}{m}\left(\frac{-1}{2}\right)\frac{4\,w^3 - 4w_0^2\,w + 8\gamma^2\,w}{\sqrt[3]{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$$

$$= \frac{-F_0}{2m}\,\frac{4\,w^3 - 4w_0^2\,w + 8\gamma^2\,w}{\sqrt[3]{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = \frac{-F_0}{2m}\,\frac{4\,w^3 - 4w_0^2\,w + 8\gamma^2\,w}{\sqrt[3]{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}] * \frac{-m}{2F_0\,w}$$

$$\frac{dA}{dw} = \frac{mF_0}{4mF_0\,w}\,\frac{4w\,(w^2 - w_0^2 + 2\gamma^2)}{\sqrt[3]{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}} = \frac{(w^2 - w_0^2 + 2\gamma^2)}{\sqrt[3]{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$$

$$\frac{dA}{dw} = 0$$

$$\therefore w^2 - w_0^2 + 2\gamma^2 = 0$$

$$w^2 = w_0^2 - 2\gamma^2$$

$$w = w_r = (w_0^2 - 2\gamma^2)^{1/2} \;\ldots\ldots (15)$$ ***Resonant Frequency Equation***

where $w_r \equiv$ ***resonant frequency*** for ***maximum amplitude***.

In case of weak damping, that is, when $c \ll 2\sqrt{mk}$ $or\ \gamma \ll w_0$

Then $w_0 \simeq w_r$

From Eq. (14) and (15) we can find A_{max} in Resonant frequency.

$$w^2 = w_0^2 - 2\gamma^2 \;\ldots\ldots (16)$$

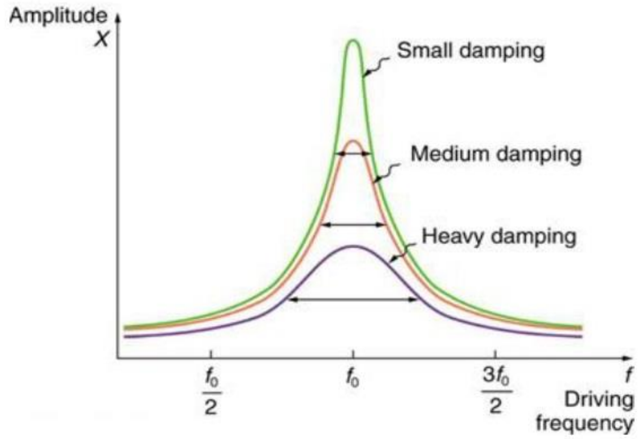
$$\therefore 2\gamma^2 = w_0^2 - w^2 \;\ldots\ldots (17)$$

Sub. Eq. (16) and (17) in Eq. (14)

$$A = \frac{F_0/m}{\sqrt{(2\gamma^2)^2 + 4\gamma^2(w_0^2 - 2\gamma^2)}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2 w_0^2 - 8\gamma^4}}$$

$$A = \frac{F_0/m}{\sqrt{4\gamma^2 w_0^2 - 4\gamma^4}}$$



Another way of designating the sharpness of the resonance peak for the driven oscillator is in terms of the parameter (Q) called ***Quality Factor*** of the resonant system.

هناك طريقة أخرى لتعيين حدة قمة الرنين للمتبذب القسري وهي من خلال حساب المعامل (Q) الذي يسمى معامل النوعية للرنين.

$$Q = \frac{w_r}{2\gamma} \ldots\ldots (23)$$

In the case of weak damping في حالة التضاؤل الضعيف

$$Q \approx \frac{w_0}{2\gamma} \ldots\ldots (24)$$

The total width Δw at the half energy points is approximately

$$\Delta w = 2\gamma \approx \frac{w_0}{Q} \ldots\ldots (25)$$

$$w = 2\pi f$$

$$\therefore \frac{\Delta w}{w_0} = \frac{\Delta f}{f_0} \simeq \frac{1}{Q} \ldots\ldots (26)$$

giving the fractional width of the resonance peak, العرض الجزئي لقمة الرنين

$Q = 10^4$ [quartz oscillators]