Chapter 4

Stresses and Strains in Statically Indeterminate Structures

Contents

4.1. Introduction

In the previous chapters, we have been discussing the cases, where simple equations of statics were sufficient to solve the examples. But, sometimes, the simple equations are not sufficient to solve such problems. Such problems are called statically indeterminate problems and the structures are called statically indeterminate structures.

For solving statically indeterminate problems, the deformation characteristics of the structure are also taken into account alongwith the statical equilibrium equations. Such equations, which contain the deformation characteristics, are called compatibility equations. The formation of such compatibility equations needs lot of patience and consideration. The solution of such statically indeterminate structures is somewhat different than the solution of simple sections and varying sections as discussed in chapters 2 and 3. So we have to adopt some indirect methods also for solving problems on statically indeterminate structures.

4.2. Types of Statically Indeterminate Structures

Though there are many types of statically indeterminate structures in the field of Strength of Materials yet the following are important from the subject point of view :

1. Simple statically indeterminate structures.
2. Indeterminate structures supporting a load.
3. Composite structures of equal lengths.
4. Composite structures of unequal lengths.

Now we shall study the procedures for the stresses and strains in the above mentioned indeterminate structures in the following pages. In order to solve the above mentioned types of statically indeterminate structures, we have to use different types of compatible equations.

4.3. Stresses in Simple Statically Indeterminate Structures

The structures in which the stresses can be obtained by forming two or more equations are called simple statically indeterminate structures. The stresses in such structures may be found out with the help of two or three compatible equations.

Example 4.1. *A square bar of 20 mm side is held between two rigid plates and loaded by an axial force P equal to 450 kN as shown in Fig. 4.1.*

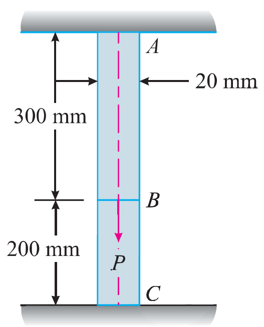

Fig. 4.1

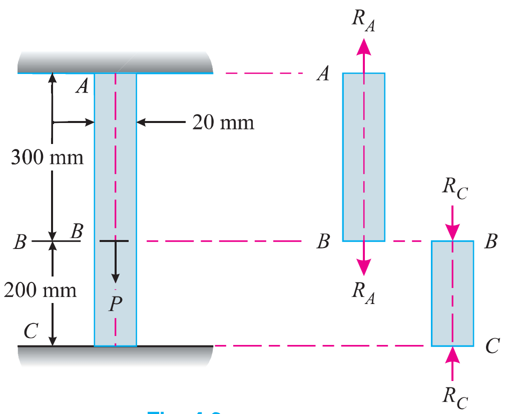

Fig. 4.2

Find the reactions at the ends A and C and the extension of the portion AB. Take E = 200 GPa.

Solution. Given : Area of bar $(A) = 20 \times 20 = 400\ \text{mm}^2$; Axial force $(P) = 450\ \text{kN} = 450 \times 10^3$ N ; Modulus of elasticity $(E) = 200\ \text{GPa} = 200 \times 10^3\ \text{N/mm}^2$; Length of AB $(l_{AB}) = 300$ mm and length of BC $(l_{BC}) = 200$ mm.

Reaction at the ends

Let R_A = Reaction at A, and

R_C = Reaction at C.

Since the bar is held between the two rigid plates A and C, therefore, the upper portion will be subjected to tension, while the lower portion will be subjected to compression as shown in Fig. 4.2.

Moreover, the increase of portion *AB* will be equal to the decrease of the portion *BC*.

We know that sum of both the reaction is equal to the axial force, *i.e.*,

$$R_A + R_C = 450 \times 10^3 \qquad ...(i)$$

Increase in the portion *AB*,

$$\delta l_{AB} = \frac{R_A\, l_{AB}}{A\,E} = \frac{R_A \times 300}{A\,E}$$

and decrease in the portion *BC*,

$$\delta l_{BC} = \frac{R_C\, l_{BC}}{A\,E} = \frac{R_C \times 200}{A\,E} \qquad ...(ii)$$

Since the value δl_{AB} is equal to that of δl_{BC}, therefore equating the equations (*ii*) and (*iii*),

$$\frac{R_A \times 300}{A\,E} = \frac{R_C \times 200}{A\,E}$$

$$R_C = \frac{R_A \times 300}{200} = 1.5\, R_A$$

Now substituting the value of R_C in equation (*ii*),

$$R_A + 1.5\, R_A = 450 \qquad \text{or} \qquad 2.5\, R_A = 450$$

$$\therefore \qquad R_A = \frac{450}{2.5} = 180 \text{ kN} \qquad \textbf{Ans.}$$

and

$$R_C = 1.5\, R_A = 1.5 \times 180 = 270 \text{ kN} \qquad \textbf{Ans.}$$

Extension of the portion AB

Substituting the value of R_A in equation (*ii*)

$$\delta_{AB} = \frac{R_A \times 300}{A\,E} = \frac{(180 \times 10^3) \times 300}{400 \times (200 \times 10^3)} = 0.675 \text{ mm} \qquad \textbf{Ans.}$$

EXAMPLE 4.2. *An aluminium bar 3 m long and 2500 mm² in cross-section is rigidly fixed at A and D as shown in Fig. 4.3.*

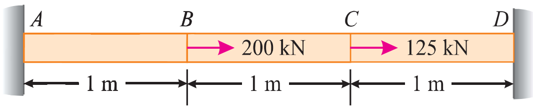

Fig. 4.3

Determine the loads shared and stresses in each portion and the distances through which the points B and C will move. Take E for aluminium as 80 GPa.

SOLUTION. Given : Total length of bar (*L*) = 3 m ; Area of cross-section A = 2500 mm² ; Modulus of elasticity (*E*) = 80 GPa = 80×10^3 N/mm² and length of portion *AB* (l_{AB}) = l_{BC} = l_{CD} = 1 m = 1×10^3 mm.

Loads shared by each portion

Let

P_{AB} = Load shared by the portion *AB*,

P_{BC} = Load shared by the portion *BC* and

P_{CD} = Load shared by the portion *CD*.

Since the bar is rigidly fixed at *A* and *D*, therefore the portion *AB* will be subjected to tension, while the portions *BC* and *CD* will be subjected to compression as shown in Fig. 4.4. Moreover, increase in the portion *AB* will be equal to the sum of the decreases in the portions *BC* and *CD*.

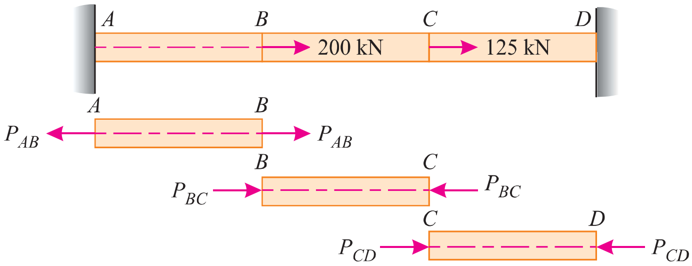

Fig. 4.4

From the geometry of the bar, we find that

$$P_{AB} + P_{BC} = 200 \quad \text{or} \quad P_{AB} = 200 - P_{BC} \quad ...(i)$$

and

$$P_{CD} - P_{BC} = 125 \quad \text{or} \quad P_{CD} = 125 + P_{BC} \quad ...(ii)$$

We know that increase in the length of portion AB,

$$\delta l_{AB} = \frac{P_{AB}\, l_{AB}}{A\,E} = \frac{P_{AB}\,(1 \times 10^3)}{A\,E} \quad ...(iii)$$

Similarly, decrease in the length of portion BC,

$$\delta l_{BC} = \frac{P_{BC}\, l_{BC}}{A\,E} = \frac{P_{BC}\,(1 \times 10^3)}{A\,E} \quad ...(iv)$$

and decrease in the length of portion CD,

$$\delta l_{CD} = \frac{P_{CD}\, l_{CD}}{A\,E} = \frac{P_{CD}\,(1 \times 10^3)}{A\,E} \quad ...(v)$$

Since the value of δl_{AB} is equal to $\delta l_{BC} + \delta l_{CD}$, therefore

$$\frac{P_{AB} \times (1 \times 10^3)}{A\,E} = \frac{P_{BC} \times (1 \times 10^3)}{A\,E} + \frac{P_{CD} \times (1 \times 10^3)}{A\,E}$$

$$\therefore \quad P_{AB} = P_{BC} + P_{CD}$$

Now substituting the values P_{AB} and P_{CD} from equations (*i*) and (*ii*) in the above equation,

$$(200 - P_{BC}) = P_{BC} + (125 + P_{BC})$$

$$\therefore \quad 3\,P_{BC} = 200 - 125 = 75 \text{ kN}$$

or

$$P_{BC} = \frac{75}{3} = 25 \text{ kN}$$

$$\therefore \quad P_{AB} = 200 - P_{BC} = 200 - 25 = 175 \text{ kN} \quad \textbf{Ans.}$$

and

$$P_{CD} = 125 + P_{BC} = 125 + 25 = 150 \text{ kN} \quad \textbf{Ans.}$$

Stresses in each portion

We know that stress in AB,

$$\sigma_{AB} = \frac{P_{AB}}{A} = \frac{175 \times 10^3}{2500} = 70 \text{ N/mm}^2 = 70 \text{ MPa (tension)} \quad \textbf{Ans.}$$

Similarly,

$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{25 \times 10^3}{2500} = 10 \text{ N/mm}^2 = 10 \text{ MPa (compression)} \quad \textbf{Ans.}$$

and $$\sigma_{CD} = \frac{P_{CD}}{A} = \frac{150 \times 10^3}{2500} = 60 \text{ N/mm}^2 = 60 \text{ MPa (compression)} \quad \textbf{Ans.}$$

Distance through which the points B and C will move

Substituting the value of P_{AB} in equation (*iii*), we get

$$\delta l_{AB} = \frac{P_{AB} \times l_{AB}}{AE} = \frac{175 \times 10^3 \times (1 \times 10^3)}{2500 \times (80 \times 10^3)} = 0.875 \text{ mm} \quad \textbf{Ans.}$$

and now substituting the value of P_{CD} in equation (*iv*), we get

$$\delta l_{CD} = \frac{P_{CD} \times l_{CD}}{AE} = \frac{(150 \times 10^3) \times (1 \times 10^3)}{2500 \times (80 \times 10^3)} = 0.75 \text{ mm} \quad \textbf{Ans.}$$

EXAMPLE 4.3. *A circular steel bar ABCD, rigidly fixed at A and D is subjected to axial loads of 50 kN and 100 kN at B and C as shown in Fig. 4.5.*

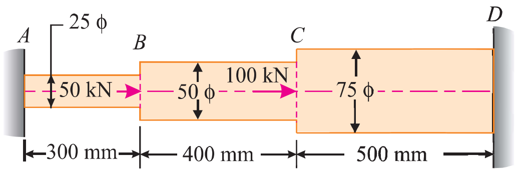

Fig. 4.5

Find the loads shared by each part of the bar and displacements of the points B and C. Take E for the steel as 200 GPa.

SOLUTION. Given : Axial load at B (P_1) = 50 kN = 50 × 10^3 N ; Axial load at C (P_2) =100 kN = 100 × 10^3 N ; Diameter of AB (D_{AB}) = 25 mm ; length of AB (l_{AB}) = 300 mm ; Diameter of BC (D_{BC}) = 50 mm ; Length of BC (l_{BC}) = 400 mm ; Diameter of CD (D_{CD}) = 75 mm ; Length of CD (l_{CD}) = 500 mm and modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2.

Loads shared by each part of the bar

Let P_{AB} = Load shared by AB,

P_{BC} = Load shared by BC, and

P_{CD} = Load shared by CD.

We know that area of the bar AB,

$$A_{AB} = \frac{\pi}{4} \times (D_{AB})^2 = \frac{\pi}{4} \times (25)^2 = 491 \text{ mm}^2$$

Similarly, area of the bar BC,

$$A_{BC} = \frac{\pi}{4} \times (D_{BC})^2 = \frac{\pi}{4} \times (50)^2 = 1964 \text{ mm}^2$$

and area of the bar CD,

$$A_{CD} = \frac{\pi}{4} \times (D_{CD})^2 = \frac{\pi}{4} (75)^2 = 4418 \text{ mm}^2$$

Since the bar is rigidly fixed at A and D, therefore, the portion AB will be subjected to tension, while the portions BC and CD will be subjected to compression as shown in Fig. 4.6. Moreover, increase in the length AB is equal to the sum of decreases in the portions BC and CD.

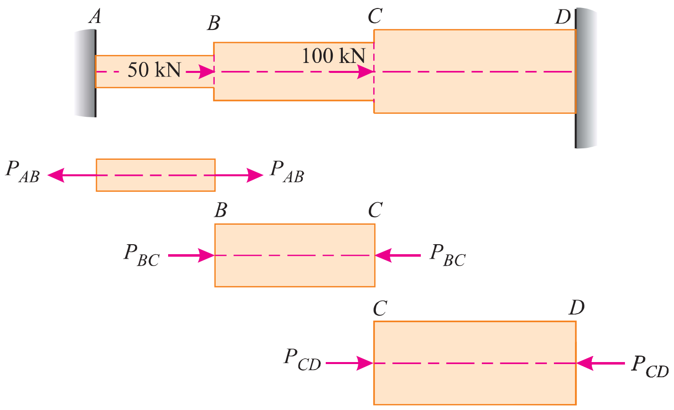

Fig. 4.6

From the geometry of the bar, we find that

$$P_{AB} + P_{BC} = 50 \quad \text{or} \quad P_{AB} = 50 - P_{BC} \qquad ...(i)$$

and $$P_{CD} - P_{BC} = 100 \quad \text{or} \quad P_{CD} = 100 + P_{BC} \qquad ...(ii)$$

We know that increase in the length of portion *AB*,

$$\delta l_{AB} = \frac{P_{AB}\, l_{AB}}{A_{AB}\, E} = \frac{P_{AB} \times 300}{491 \times (200 \times 10^3)} = 3.05 \times 10^{-6}\, P_{AB} \text{ mm} \qquad ...(iii)$$

Similarly, $$\delta l_{BC} = \frac{P_{BC}\, l_{BC}}{A_{BC}\, E} = \frac{P_{BC} \times 400}{1964 \times (200 \times 10^3)} = 1.02 \times 10^{-6}\, P_{BC} \text{ mm} \qquad ...(iv)$$

and $$\delta l_{CD} = \frac{P_{CD}\, l_{CD}}{A_{CD}\, E} = \frac{P_{CD} \times 500}{4418 \times (200 \times 10^3)} = 0.57 \times 10^{-6}\, P_{CD} \text{ mm} \qquad ...(v)$$

Since the value of δl_{AB} is equal to $\delta l_{BC} + \delta l_{CD}$, therefore

$$3.05 \times 10^{-6}\, P_{AB} = 1.02 \times 10^{-6}\, P_{BC} + 0.57 \times 10^{-6}\, P_{CD}$$

$$\therefore \quad 305\, P_{AB} = 102\, P_{BC} + 57\, P_{CD}$$

Now substituting the values of P_{AB} and P_{CD} from equations (*i*) and (*ii*) in the above equation,

$$305\,(50 - P_{BC}) = 102\, P_{BC} + 57\,(100 + P_{BC})$$

$$15\,250 - 305\, P_{BC} = 102\, P_{BC} + 5700 + 57\, P_{BC}$$

$$\therefore \quad 464\, P_{BC} = 9\,550 \quad \text{or} \quad P_{BC} = \frac{9550}{464} = 20.6 \text{ kN} \qquad \textbf{Ans.}$$

Similarly, $$P_{AB} = 50 - P_{BC} = 50 - 20.6 = 29.4 \text{ kN} \qquad \textbf{Ans.}$$

and $$P_{CD} = 100 + P_{BC} = 100 + 20.6 = 120.6 \text{ kN} \qquad \textbf{Ans.}$$

Displacements of the points B and C

Now substituting the value of P_{AB} in equation (*iii*), we get

$$\delta l_{AB} = 3.05 \times 10^{-6}\, P_{AB} = 3.05 \times 10^{-6} \times (29.4 \times 10^3) = 0.90 \text{ mm} \qquad \textbf{Ans.}$$

and now substituting the value of P_{CD} in equation (*v*), we get

$$\delta l_{CD} = 0.57 \times 10^{-6} \times P_{CD} = 0.57 \times 10^{-6} \times (120.6 \times 10^3) = 0.07 \text{ mm} \quad \textbf{Ans.}$$

EXERCISE 4.1

1. An alloy bar 800 mm long and 200 mm^2 in cross-section is held between two rigid plates and is subjected to an axial load of 200 kN as shown in Fig. 4.7.

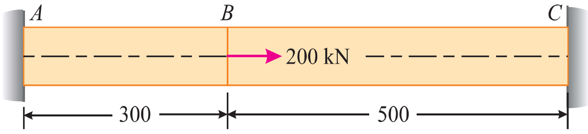

Fig. 4.7

Find the reactions at the two ends *A* and *C* as well as extension of the portion *AB*.

[**Ans.** 125 kN ; 75 kN ; 0.094 mm]

2. A bar *ABC* fixed at both ends *A* and *C* is loaded by an axial load (*P*) at *C*. If the distances *AB* and *BC* are equal to *a* and *b* respectively then find the reactions at the ends *A* and *C*.

3. An axial force of 20 kN is applied to a steel bar *ABC* which is fixed at both ends *A* and *C* as shown in Fig. 4.8.

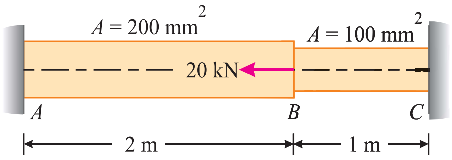

Fig. 4.8

Determine the reactions at both the supports and stresses developed in two parts of the bar. Take E = 200 GPa. [**Ans.** $R_A = R_C$ = 10 kN ; σ_{AB} = 50 MPa (*C*); σ_{BC} = 100 MPa (*T*)]

4. A prismatic bar *ABCD* has built-in ends *A* and *D*. It is subjected to two point loads P_1 and P_2 equal to 80 kN and 40 kN at *B* and *C* as shown in Fig. 4.9.

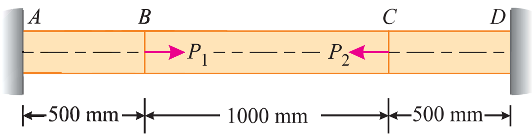

Fig. 4.9

Find the reactions at *A* and *D*. [**Ans.** 70 kN ; 50 kN]

4.4. Stresses in Indeterminate Structures Supporting a Load

Sometimes, we come across a set of two or more members supporting a load. In such cases, the deformation of all the members will be the same. If the members are of different cross-sections or have different modulus of elasticity, then the stresses developed in all the members will be different.

EXAMPLE 4.4. *A block shown in Fig. 4.10 weighing 35 kN is supported by three wires. The outer two wires are of steel and have an area of 100* mm^2 *each, whereas the middle wire of aluminium and has an area of 200* mm^2*.*

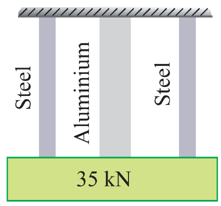

Fig. 4.10

If the elastic modulii of steel and aluminium are 200 GPa and 80 GPa respectively, then calculate the stresses in the aluminium and steel wires.

SOLUTION. Given: Total load (P) = 35 kN = 35 × 10^3 N ; Total area of steel rods (A_S) = 2 × 100 = 200 mm^2 ; Area of aluminium rod (A_A) = 200 mm^2 ; Modulus of elasticity of steel (E_S) = 200 GPa = 200 × 10^3 N/mm^2; Modulus of elasticity of aluminium (E_A) = 80 GPa = 80 × 10^3 N/mm^2 and load supported by wires (P) = 35 kN = 35 × 10^3 N

Let σ_S = Stress in steel wires,

σ_A = Stress in aluminium wire and

l = Length of the wires.

We know that increase in the length of steel wires,

$$\delta l_S = \frac{\sigma_S \times l_S}{E_S} = \frac{\sigma_S \times l}{200 \times 10^3}$$

Similarly,
$$\delta l_A = \frac{\sigma_A \times l_A}{E_A} = \frac{\sigma_A \times l}{80 \times 10^3}$$

Since increase in the lengths of steel and aluminium wires is equal, therefore equating equations (*i*) and (*ii*), we get

$$\frac{\sigma_S \times l}{200 \times 10^3} = \frac{\sigma_A \times l}{80 \times 10^3} \quad \text{or} \quad \sigma_S = \frac{200}{80} \times \sigma_A = 2.5\,\sigma_A$$

We also know that load supported by the three wires (P),

$$35 \times 10^3 = (\sigma_S \,.\, A_S) + (\sigma_A \,.\, A_A) = (2.5\,\sigma_A \times 200) + (\sigma_A \times 200) = 700\,\sigma_A$$

$$\therefore \quad \sigma_A = \frac{35 \times 10^3}{700} = 50 \text{ N/mm}^2 = 50 \text{ MPa} \quad \textbf{Ans.}$$

and
$$\sigma_S = 2.5\,\sigma_A = 2.5 \times 50 = 125 \text{ MPa} \quad \textbf{Ans.}$$

EXAMPLE 4.5. *A steel rod of cross-sectional area 800 mm^2 and two brass rods each of cross-sectional area 500 mm^2 together support a load of 25 kN as shown in Fig. 4.11.*

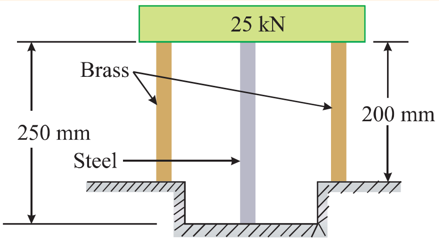

Fig. 4.11

Calculate the stresses in the rods. Take E for steel as 200 GPa and E for brass as 100 GPa.

SOLUTION. Given : Area of one steel rod, (A_S) = 800 mm^2 ; Total Area of two brass rods (A_B) = 2 × 500 = 1000 mm^2 ; Total load (P) = 25 kN = 25 × 10^3 N ; Modulus of elasticity of steel (E_S) = 200 GPa : Modulus of elasticity of brass (E_B) = 100 GPa ; Length of steel bar (l_S) = 250 mm and length of brass rod (l_B) = 200 mm.

Let σ_S = Stress in steel rod and

σ_B = Stress in brass rod.

We know that decrease in the length of the steel rod due to stress,

$$\delta l_S = \frac{\sigma_S\, l_S}{E_S} = \frac{\sigma_S \times 250}{200 \times 10^3} = 1.25 \times 10^{-3}\, \sigma_S$$

and decrease in the length of the brass rods due to stress,

$$\delta l_B = \frac{\sigma_B\, l_B}{E_B} = \frac{\sigma_B \times 200}{100 \times 10^3} = 2 \times 10^{-3}\, \sigma_B$$

Since the value of δl_S is equal to that of δl_B, therefore equating equations (*i*) and (*ii*), we get

$$1.25 \times 10^{-3}\, \sigma_S = 2 \times 10^{-3}\, \sigma_B \qquad \text{or} \qquad \sigma_S = \frac{2}{1.25} \times \sigma_B = 1.6\, \sigma_B$$

We also know that total load shared by all the three rods (P),

$$25 \times 10^3 = \sigma_S A_S + \sigma_B A_B = (1.6\, \sigma_B \times 800) + (\sigma_B \times 1000) = 2280\, \sigma_B$$

$$\therefore \qquad \sigma_B = \frac{25 \times 10^3}{2280} = 11.0 \text{ N/mm}^2 = 11.0 \text{ MPa} \qquad \textbf{Ans.}$$

and $\sigma_S = 1.6\, \sigma_B = 1.6 \times 11.0 = 17.6$ MPa **Ans.**

EXAMPLE 4.6. *A load of 80 kN is jointly supported by three rods of 20 mm diameter as shown in Fig. 4.12.*

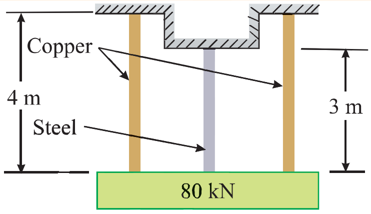

Fig. 4.12

The rods are adjusted in such a way that they share the load equally. If an additional load of 50 kN is added, find the final stresses in steel and copper. Take E for copper as 100 GPa and for steel as 200 GPa.

SOLUTION. Given : Total load (P_1) = 80 kN = 80×10^3 N ; Diameter of each rod (d) = 20 mm ; Additional load (P_2) = 50 kN = 50×10^3 N ; Modulus of elasticity of copper (E_C) = 100 GPa = 100×10^3 N/mm^2 and modulus of elasticity of steel (E_S) = 200 GPa = 200×10^3 N/mm^2.

We know that total area of two copper rods

$$A_C = 2 \times \frac{\pi}{4} \times (d)^2 = 2 \times \frac{\pi}{4} \times (20)^2 = 200\, \pi \text{ mm}^2$$

and area of one steel rod

$$A_S = \frac{\pi}{4} \times (d)^2 = \frac{\pi}{4} \times (20)^2 = 100\, \pi \text{ mm}^2$$

First of all consider the 80 kN load only, which is shared equally by all the three rods. We know that initial stress in each rod

$$= \frac{80 \times 10^3}{3 \times 100\pi} = 84.9 \text{ N/mm}^2 = 84.9 \text{ MPa} \qquad ...(i)$$

Now consider an additional load of 50 kN, which is added to the existing load of 80 kN. We know that this additional load will cause some additional stresses in all the three rods.

Let σ_C = Additional stress in copper rods, and

σ_S = Additional stress in steel rod

We know that increase in the length of copper rods due to stress,

$$\delta l_C = \frac{\sigma_C \times l_C}{E_C} = \frac{\sigma_C \times (4 \times 10^3)}{100 \times 10^3} = 0.04\ \sigma_C \qquad ...(ii)$$

and increase in the length of steel rod due to stress,

$$\delta l_S = \frac{\sigma_S \times l_S}{E_S} = \frac{\sigma_S \times (3 \times 10^3)}{200 \times 10^3} = 0.015\ \sigma_S \qquad ...(iii)$$

Since the value of δl_C is equal to that of δl_S, therefore equating the equations (*ii*) and (*iii*)

$$0.04\ \sigma_C = 0.015\ \sigma_S \quad \text{or} \quad \sigma_C = 0.375\ \sigma_S$$

We also know that additional load supported by the three rods (P_2)

$$50 \times 10^3 = (\sigma_S \cdot A_S) + (\sigma_C \cdot A_C) = (\sigma_S \times 100\ \pi) + (0.375\ \sigma_S \times 200\ \pi)$$

$$= 175\ \pi\ \sigma_S$$

or $$\sigma_S = \frac{50 \times 10^3}{175\pi} = 90.9 \text{ N/mm}^2 = 90.9 \text{ MPa}$$

and $$\sigma_C = 0.375\ \sigma_S = 0.375 \times 90.9 = 34.1 \text{ MPa}$$

∴ Final stress in the steel

= 84.9 + 90.9 = 175.8 MPa **Ans.**

and final stress in copper = 84.9 + 34.1 = 119.0 MPa **Ans.**

EXAMPLE 4.7. *Two vertical rods one of steel and the other of copper are rigidly fastened at their upper end at a horizontal distance of 200 mm as shown in Fig. 4.13.*

The lower ends support a rigid horizontal bar, which carries a load of 10 kN. Both the rods are 2.5 m long and have cross-sectional area of 12.5 mm². Where should a load of 10 kN be placed on the bar, so that it remains horizontal after loading? Also find the stresses in each rod. Take E_S *= 200 GPa and* E_C *= 110 GPa. Neglect bending of the cross-bar.*

SOLUTION. Given : Distance between the bars = 200 mm ; Total load (P) = 10 kN = 10×10^3 N ; Length of steel rod (l_S) = l_C = 2.5 m = 2.5×10^3 mm ; Area of steel rod (A_S) = A_C = 12.5 mm²; Modulus of elasticity of steel (E_S) = 200 GPa = 200×10^3 N/mm² and modulus of elasticity of copper (E_C) = 110 GPa = 110×10^3 N/mm².

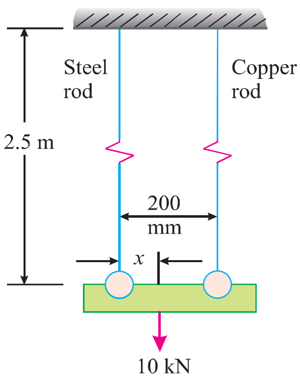

Fig. 4.13

Position of the load

Let x = Distance between the load and steel rod in mm

As a matter of fact, the load of 10 kN will be shared by both the rods in such a way that they cause equal extension.

Let P_S = Load shared by the steel rod, and

P_C = Load shared by the copper rod.

$\therefore \quad P_S + P_C = 10 \text{ kN}$ ...(*i*)

We know that extension of the steel rod,

$$\delta l_S = \frac{P_S\, l}{A_S\, E_S} = \frac{P_S \times (2.5 \times 10^3)}{12.5 \times (200 \times 10^3)} = \frac{P_S}{1000} \qquad ...(ii)$$

and extension of the copper rod,

$$\delta l_C = \frac{P_C\, l}{A_C\, E_C} = \frac{P_C \times (2.5 \times 10^3)}{12.5 \times (110 \times 10^3)} = \frac{P_C}{550} \qquad ...(iii)$$

Since both the extensions are equal, therefore equating equations (*ii*) and (*iii*)

$$\frac{P_S}{1000} = \frac{P_C}{550} \quad \text{or} \quad \frac{P_S}{P_C} = \frac{1000}{550} = \frac{20}{11} \qquad (iv)$$

Now taking moments of the loads about the steel bar and equating the same,

$$10 \times x = P_C \times 200 \quad \text{or} \quad (P_S + P_C)\, x = 200\, P_C$$

$$P_S.x + P_C.x = 200\, P_C \quad \text{or} \quad P_S.x = 200\, P_C - P_C.x = P_C\,(200 - x)$$

$$\therefore \quad \frac{P_S}{P_C} = \frac{200 - x}{x} \qquad ...(v)$$

Now equating two values of $\frac{P_S}{P_C}$ from equations (*iv*) and (*v*),

$$\frac{20}{11} = \frac{200 - x}{x} \quad \text{or} \quad 20x = 2200 - 11x$$

$$\therefore \quad 31x = 2200 \quad \text{or} \quad x = \frac{2200}{31} = 71 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in each rod

From equation (*iv*), we find that

$$\frac{P_S}{P_C} = \frac{20}{11} \quad \text{or} \quad 11\, P_S - 20\, P_C - 20\,(10 - P_S) = 200 - 20\, P_S$$

$$\therefore \quad 31\, P_S = 200 \quad \text{or} \quad P_S = \frac{200}{31} = 6.45 \text{ kN} = 6.45 \times 10^3 \text{ N}$$

and $\quad P_C = 10 - P_S = 10 - 6.45 = 3.5 \text{ kN} = 3.5 \times 10^3 \text{ N}$

$\therefore$ Stress in steel rod,

$$\sigma_S = \frac{P_S}{A_S} = \frac{6.45 \times 10^3}{12.5} = 516 \text{ N/mm}^2 = 516 \text{ MPa} \qquad \textbf{Ans.}$$

and stress in copper rod,

$$\sigma_C = \frac{P_C}{A_C} = \frac{3.5 \times 10^3}{12.5} = 280 \text{ N/mm}^2 = 280 \text{ MPa} \qquad \textbf{Ans.}$$

EXAMPLE 4.8. *A load of 5 kN is suspended by ropes as shown in Fig. 4.14 (a) and (b). In both the cases, the cross-sectional area of the ropes is 200 mm² and the value of E is 1.0 GPa.*

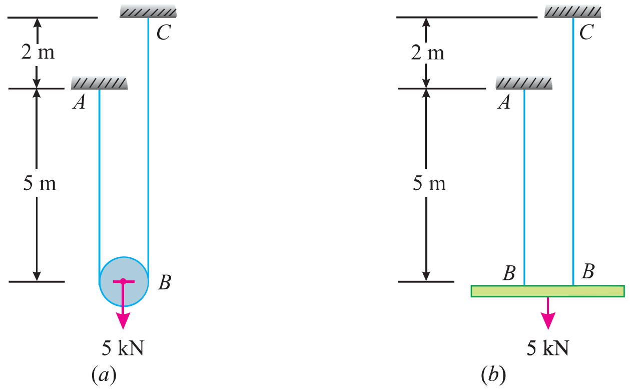

Fig. 4.14

In (a) the rope ABC is continuous over a smooth pulley, from which the load is suspended. In (b) the ropes AB and CB are separate ropes joined to a block, from which the load is suspended in such a way, that both the ropes are stretched by the same amount. Determine, for both the cases, stresses in the ropes and the deflections of the pulley and the block due to the load.

SOLUTION. Given : Total load $(P) = 5 \text{ kN} = 5 \times 10^3$ N ; Length of AB $(l_{AB}) = 5 \text{ m} = 5 \times 10^3$ mm; Length of BC $(l_{BC}) = 7 \text{ m} = 7 \times 10^3$ mm ; Area of each rope $(A) = 200 \text{ mm}^2$ and modulus of elasticity $(E) = 1.0 \text{ GPa} = 1.0 \times 10^3 \text{ N/mm}^2$.

First case

We know that the load of 5 kN is suspended from the pulley, therefore load shared by both the ropes is equal. Or in other words, load shared by each rope.

$$P_1 = \frac{5 \times 10^3}{2} = 2.5 \times 10^3 \text{ N}$$

∴ Stress in the ropes,

$$\sigma = \frac{P_1}{A} = \frac{2.5 \times 10^3}{200} = 12.5 \text{ N/mm}^2 = 12.5 \text{ MPa} \quad \textbf{Ans.}$$

and total elongation of the rope ABC,

$$\delta l = \frac{P_1 l_{AB}}{AE} + \frac{P_1 l_{BC}}{AE}$$

$$= \frac{(2.5 \times 10^3) \times (5 \times 10^3)}{200 \times (1.0 \times 10^3)} + \frac{(2.5 \times 10^3) \times (7 \times 10^3)}{200 \times (1.0 \times 10^3)}$$

$$= 62.5 + 87.5 = 150 \text{ mm}$$

∴ Deflection of the pulley

$$= \frac{150}{2} = 75 \text{ mm} \quad \textbf{Ans.}$$

Second case

Let σ_{AB} = Stress in the rope AB, and

σ_{BC} = Stress in the rope BC.

We know that deflection of the rope AB,

$$\delta l_{AB} = \frac{\sigma_{AB} \cdot l_{AB}}{E} = \frac{\sigma_{AB} \times (5 \times 10^3)}{1 \times 10^3} = 5\,\sigma_{AB}$$

and deflection of the rope *BC*,

$$\delta l_{BC} = \frac{\sigma_{BC} \cdot l_{BC}}{E} = \frac{\sigma_{BC} \times (5 \times 10^3)}{1 \times 10^3} = 7\,\sigma_{AB}$$

Since both the deflections are equal, therefore equating the value of (*i*) and (*ii*),

$$5\,\sigma_{AB} = 7\,\sigma_{BC} \qquad \text{or} \qquad \sigma_{AB} = \frac{7}{5} \times \sigma_{BC}$$

We also know that the load (*P*) of 5 kN is shared by both the ropes, therefore load (*P*)

$$5 \times 10^3 = \sigma_{AB} \times A + \sigma_{BC} \times A = \left(\frac{7}{5} \times \sigma_{BC} \times 200\right) + (\sigma_{BC} \times 200)$$

$$= 480\,\sigma_{BC}$$

$$\therefore \qquad \sigma_{BC} = \frac{5 \times 10^3}{480} = 10.4 \text{ N/mm}^2 = 10.4 \text{ MPa} \qquad \textbf{Ans.}$$

and

$$\sigma_{AB} = \frac{7}{5}\,\sigma_{BC} = \frac{7}{5} \times 10.4 \;\; 14.56 \text{ MPa} \qquad \textbf{Ans.}$$

Now substituting the value of σ_{AB} is equation (*i*),

$$\delta l_{AB} = 5\,\sigma_{AB} = 5 \times 14.56 = 72.8 \text{ mm} \qquad \textbf{Ans.}$$

NOTE. The deflection of the block may also be found out by equating the value of σ_{BC} in equation (*ii*),

$$\delta l_{BC} = 7\,\sigma_{BC} = 7 \times 10.4 = 72.8 \text{ mm} \qquad \textbf{Ans.}$$

EXERCISE 4.2

1. Three long parallel wires equal in length are supporting a rigid bar connected at their bottoms as shown in Fig. 4.15. If the cross-sectional area of each wire is 100 mm^2, calculate the stresses in each wire. Take E_B = 100 GPa and E_S = 200 GPa. [**Ans.** σ_B = 25 MPa ; σ_S = 50 MPa]

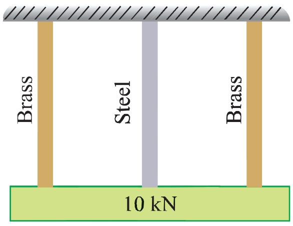

Fig. 4.15

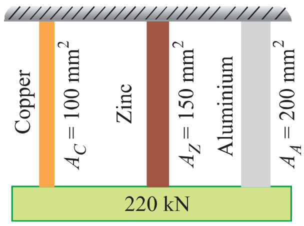

Fig. 4.16

2. Three wires made of copper, zinc and aluminium are of equal lengths and have cross-sectional areas of 100, 150 and 200 square mm respectively. They are rigidly connected at their ends as shown in Fig. 4.16. If this compound member is subjected to a longitudinal pull of 220 kN, estimate the load carried on each wire. Take E_C = 130 GPa, E_Z = 100 GPa and E_A = 80 GPa. [**Ans.** P_C = 65 kN, P_Z = 75 kN, P_A = 80 kN]

3. Two steel rods and one copper rod each of 20 mm diameter together support a load of 50 kN as shown in Fig.4.17. Find the stresses in each rod. Take *E* for steel and copper as 200 GPa and 100 GPa respectively. [**Ans.** σ_C = 39.8 MPa ; σ_S = 59.7 MPa]

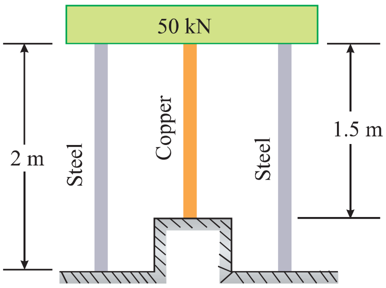

Fig. 4.17

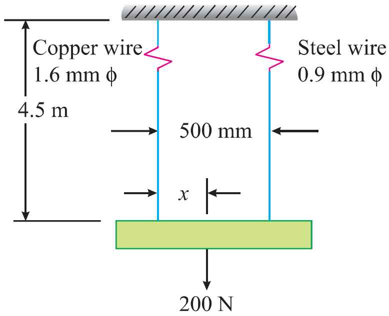

Fig. 4.18

4. Two vertical wires are suspended at a distance of 500 mm apart as shown in Fig. 4.18. Their upper ends are firmly secured and their lower ends support a rigid horizontal bar, which carries a load of 200 N. The left hand wire has a diameter of 1.6 mm and is made of copper, and the right hand wire has a diameter of 0.9 mm and is made of steel. Both wires, initially, are 4.5 metres long. Determine :

(*a*) Position of the line of action of the load, if both the wires extend by the same amount.

(*b*) Slope of the rigid wire, if the load is hung at the centre of the bar. Neglect weight of the bar.

Take E for copper as 100 GPa and E for steel as 200 GPa. [**Ans.** 170 mm ; 0.15°]

4.5. Stresses in Composite Structures of Equal Lengths

We have already discussed in Art 3.6 the procedure for stresses in the bars of composite sections. The same principle can be extended to the statically indeterminate structures also. Though there are many types of such structures, yet a rod passing axially through a pipe is an important structure from the subject point of view.

EXAMPLE 4.9. *A mild steel rod of 20 mm diameter and 300 mm long is enclosed centrally inside a hollow copper tube of external diameter 30 mm and internal diameter 25 mm. The ends of the rod and tube are brazed together, and the composite bar is subjected to an axial pull of 40 kN as shown in Fig. 4.19.*

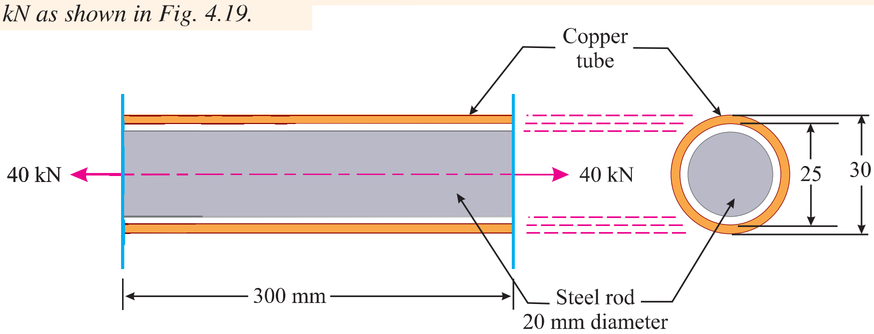

Fig. 4.19

If E for steel and copper is 200 GPa and 100 GPa respectively, find the stresses developed in the rod and the tube.

SOLUTION. Given : Diameter of steel rod = 20 mm ; External diameter of copper tube = 30 mm; Internal diameter of copper tube = 25 mm ; Total load (P) = 40 kN = 40×10^3 N ; Modulus of elasticity of steel (E_S) = 200 GPa and modulus of elasticity of copper (E_C) = 100 GPa.

Let σ_S = Stress developed in the steel rod and

σ_C = Stress developed in the copper tube.