Al- Mustaqbal University

College of Science

Medical Physics Department

First Stage

Mechanics

Lecture (1): Oscillations

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2024 – 2025

Introduction

Everywhere around us we see systems engaged in a ***periodic motion***: the small *oscillations* of a pendulum clock, a child playing on a *swing*, the rise and fall of the *tides*, the *swaying* of a tree in the wind and the *vibrations* of the strings on a violin. The essential feature that all these phenomena have in common is ***periodicity***, a pattern of movement or displacement that repeats itself over and over again.

في كل مكان حولنا نرى أنظمة تعمل بحركة دورية: التذبذبات الصغيرة لساعة البندول و تأرجح الطفل وهو يلعب على أرجوحة وصعود وانحدار المد والجزر وتمايل الاشجار في الريح واهتزازات أوتار الكمان. الميزة الأساسية التي تشترك فيها كل هذه الظواهر هي الدورية، والتي هي نمط من الحركة أو الازاحة الذي يعيد نفسه مرارًا وتكرارًا.

Linear Restoring Force, Harmonic Motion

One of the most important cases of rectilinear motion is that produced by a ***linear restoring force***. This force whose magnitude is **proportional** to the **displacement** of the particle from the **equilibrium position** and whose **direction** is always **opposite** to that of the displacement. Such a force is exerted by a spring obeying Hooke’s law.

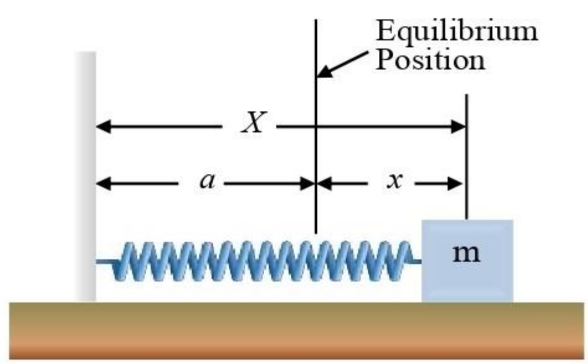
واحدة من أهم حالات الحركة على خط مستقيم تلك التي تحدثها قوة معيدة خطية. هذه القوة التي يتناسب مقدارها مع إزاحة الجسم من موضع التوازن اتجاهها يكون دائماً عكس اتجاه الازاحة. قوة كهذه يسببها وتر أو نابض يخضع لقانون هوك.

$x = X - a \quad \ldots\ldots (1)$

Eq. (1) represents the displacement of the spring from its equilibrium length.

X is total length

a is upstretched (zero) length of the spring



Elastic force exerted by a spring obeying ***Hooke's law***

$F \; \alpha - x$

$F = -kx$ ……(2) ***Restoring Force (Hooke's law)***

where k is called ***Stiffness (Spring Constant)*** معامل المرونة

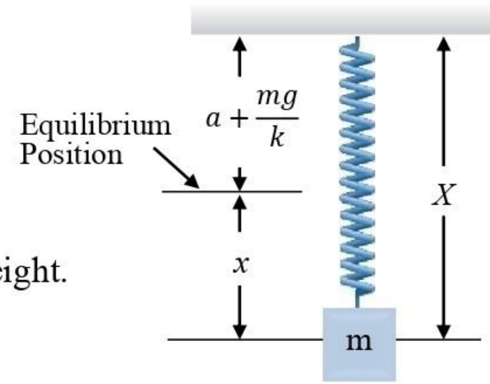
Sub. Eq.(1) in Eq.(2)

$\therefore F = -k(X - a) \dots \dots (3)$

Let the same spring be held vertically. The total force acting on the particle is:

$F = -kx$

$F = -k(X - a) + mg \dots \dots (4)$



where the *positive* direction is downward:

$x = X - \left(a + \frac{mg}{k}\right)$

$\frac{mg}{k}$ is the change in displacement due to body weight.

$\therefore x = X - a - \frac{mg}{k}$

$X - a = x + \frac{mg}{k} \dots \dots (5)$

Sub. Eq. (5) in Eq. (4)

$\therefore F = -k\left(x + \frac{mg}{k}\right) + mg$

$F = -kx - \frac{kmg}{k} + mg$

This give again:

$\therefore F = -kx$

$-kx = m\ddot{x}$

$\therefore m\ddot{x} + kx = 0 \dots \dots (6)$

Eq. (6) is **second order** differential Eq. with constant coefficients of the harmonic oscillator or linear oscillator. As equation $y'' + py' + qy = 0$, so, to solve such equation we shall employ the ***trail method*** in which the function (Ae^{qt}) is the trail solution, where (q) is a constant to be determined.

معادلة (6) هي معادلة تفاضلية من الدرجة الثانية ذات معاملات ثابتة لمذبذب توافقي أو مذبذب خطي. ويتم استخدام طريقة التجربة لحل المعادلة والذي فيه نجرب الحل (Ae^{qt}) حيث ان (q) ثابت.

$$x = Ae^{qt} \dots\dots (7)$$

$$\frac{dx}{dt} = \dot{x} = Aqe^{qt} \dots\dots (8)$$

$$\frac{d^2x}{dt^2} = \ddot{x} = A\,q^2\,e^{qt} \dots\dots (9)$$

Sub. Eq. (7) and (9) in Eq. (6)

$$m\,q^2\,A\,e^{qt} + kA\,q^{qt} = 0 \quad] \div A\,e^{qt}$$

$$mq^2 + k = 0$$

$$mq^2 = -k$$

$$q^2 = -\frac{k}{m}$$

$$\therefore q = \mp\sqrt{\frac{-k}{m}} = \mp i\sqrt{\frac{k}{m}}$$

where $i = \sqrt{-1}$, $\sqrt{\frac{k}{m}} = w_0$

$$q = \mp iw_0 \dots\dots (10)$$

Sub. Eq. (10) in Eq. (7)

$$\therefore x = Ae^{\mp i\,w_0 t} \dots\dots (11)$$

For a linear differential eqns., solution are additive, so, the general solution is:

$$x = A_+\,e^{iw_0 t} + A_-\,e^{-iw_0 t} \dots\dots (12)$$

using ***Euler's formula***

$$e^{iu} = \cos u + i \sin u$$

So we can rewrite Eq. (12) in the form:

$$x = A_+(\cos w_0 t + i \sin w_0 t) + A_-(\cos w_0 t - i \sin w_0 t)$$

$$= A_+ \cos w_0 t + i\,A_+ \sin w_0 t + A_- \cos w_0 t - i\,A_- \sin w_0 t$$

$$= (A_+ + A_-) \cos w_0 t + (iA_+ - iA_-) sin w_0 t$$

$$x = a \sin w_0 t + b \cos w_0 t \dots\dots (13)$$

where $a = i\,A_+ - i\,A_-$ and $b = A_+ + A_-$

The real solution of Eq. (13) is:

$$x = b \cos w_0 t$$

or $x = A\cos(w_0 t + \theta_o) \dots\dots (14)$ ***Sinusoidal Oscillation of Displacement x*** where: w_o is ***angular frequency*** التردد الزاوي

A is ***amplitude*** (the maximum value of x) السعة (اعظم قيمة للازاحة)

Equation (14) represent cosine function for two angle

So, $A\cos(w_0 t + \theta_o) = A(\cos w_0 t \cos\theta_o + \sin w_0 t \sin\theta_o)$

Let $a = A\cos\theta_o$, $b = A\sin\theta_o$

$$a^2 + b^2 = (A\cos\theta_o)^2 + (A\sin\theta_o)^2 = A^2(\cos^2\theta_o + \sin^2\theta_o)$$

$$A = (a^2 + b^2)^{1/2}$$

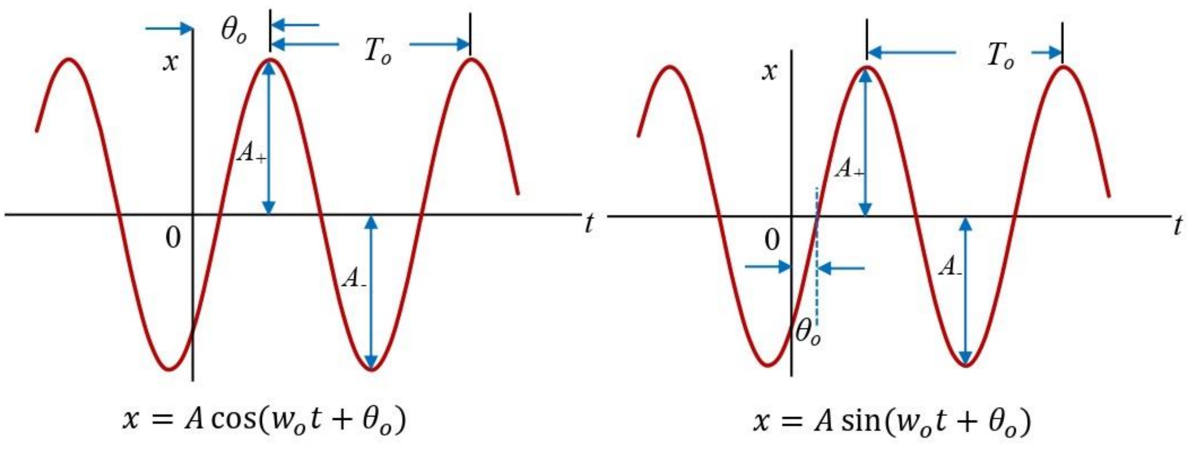
$$\frac{b}{a} = \frac{A\sin\theta_o}{A\cos\theta_o} = \tan\theta_o$$

$$\theta_0 = \tan^{-1}\left(\frac{b}{a}\right) \dots\dots(15)$$ ***Initial Phase***

T_o is **time period** of the oscillation (time required for one complete cycle); that is, the period is the time for which the product (wt) increase by just (2π)

$$T_o = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}} \dots\dots(16)$$ زمن الذبذبة الزمن اللازم لدورة كاملة

$$w_0 = 2\pi f_0 \dots\dots(17)$$



$f_0 \equiv$ Linear frequency of oscillation (is the number of cycles in unit time)

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \dots\dots(18)$$ التردد الخطي للمتذبذب والذي يمثل عدد الدورات لوحدة الزمن

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Example:

A light spring is found to stretch an amount b when it supports a block of mass m. If the block is pulled downward a distance l from its equilibrium position and released at time $= 0$, find the resulting motion as a function of t.

Solution:

In the static equilibrium

$$F = -kb = -mg$$

$$\therefore k = \frac{mg}{b}$$

$$w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mb}} = \sqrt{\frac{g}{b}}$$

Now find the constants for the equation of motion

$$x = A\cos(w_0 t + \theta_0)$$

$$at\ \ t = 0\ , \quad x = l \qquad \text{and } \dot{x} = 0$$

$$\dot{x} = -A\, w_0 \sin(w_0 t + \theta_0)$$

$$\dot{x} = -A\, w_0 \sin(w_0(0) + \theta_0) = 0$$

$$A\, w_0 \neq 0\ ,\ \ A = l\ \ is\ spring\ length, w_0\ is\ angular\ frequncy$$

$$A = l = X_{max}$$

$$\therefore \sin(\theta_0) = 0 \ \ \Rightarrow\ \ \theta_0 = 0$$

$$x = A\cos(w_0 t)$$

$$\therefore x = l\cos\left(\sqrt{\frac{g}{b}}\, t\right)$$

Energy Consideration in Harmonic Motion

Consider a particle moving under a linear restoring force$= -kx$. Let us calculate the work done by an external force f_a in moving the particle from the equilibrium position $x = 0$ to some position x.

نفترض جسيم يتحرك تحت تأثيرقوة معيدة خطية $-kx$ نحسب الشغل المنجز بواسطة قوة خارجية f_a لنقل الجسم من موضع التوازن $x = 0$ إلى موضع ما x

$$F = -kx \dots\dots (1)$$

$$F_a = -F = kx$$

$$\therefore W = \int F_a \, dx = \int_0^x kx \, dx = \frac{1}{2}kx^2$$

The work W is stored in the spring as potential energy الشغل يخزن في النابض كطاقة كامنة

$$\therefore V(x) = W = \frac{1}{2}kx^2 = E_p \dots\dots (2)$$

Total spring energy

$$E = E_k + E_p$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \dots\dots (3) \qquad * \frac{2}{m}$$

$$\frac{2E}{m} = \dot{x}^2 + \frac{k}{m}x^2$$

$$\dot{x}^2 = \frac{2E}{m} - \frac{k}{m}x^2$$

$$\dot{x} = \left(\frac{2E}{m} - \frac{k}{m}x^2\right)^{1/2} \dots\dots (4)$$

This can be integrated to give (t) as function of x

$$\dot{x} = \frac{dx}{dt} \quad \Rightarrow dt = \frac{dx}{\dot{x}}$$

$$t = \int dt = \int \frac{dx}{\left(\frac{2E}{m} - \frac{k}{m}x^2\right)^{1/2}} \dots\dots (5)$$

➢ ***Derive the Equation of Time***

• ***When $x = A\cos\theta$***

Rewrite Eq.(5)

$$t = \int \frac{dx}{\left(\frac{k}{m}\right)\left[\frac{2E}{k} - x^2\right]^{1/2}}$$

$$= \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} = \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{A^2 - x^2}} \dots\dots (6)$$

where $A = \sqrt{\frac{2E}{k}} \equiv$ amplitude

$$x = A\cos\theta \dots\dots (7)$$

$$\therefore x^2 = A^2 \cos^2\theta$$

$$\therefore A^2 - x^2 = A^2 - A^2\cos^2\theta$$

$$= A^2(1-\cos^2\theta)$$

$$A^2 - x^2 = A^2\sin^2\theta$$

$$\sqrt{A^2-x^2} = A\sin\theta \;\ldots\ldots (8)$$

From Eq.(7)

$$dx = -A\sin\theta\, d\theta \;\ldots\ldots (9)$$

Sub. Eqns.(8) and (9) in Eq. (6)

$$t = -\sqrt{\frac{m}{k}}\int \frac{A\sin\theta}{A\sin\theta}\, d\theta$$

$$t = -\sqrt{\frac{m}{k}}\int d\theta$$

$$t = -\sqrt{\frac{m}{k}}\theta + c \;\ldots\ldots (10)$$

$$\because\; x = A\cos\theta$$

$$\therefore \cos\theta = \frac{x}{A} \Rightarrow \theta = \cos^{-1}\left(\frac{x}{A}\right) \;\ldots\ldots (11)$$

Sub. Eq.(11) in Eq. (10)

$$t = -\sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{x}{A}\right) + c \;\ldots\ldots (12)$$

- ***When*** $\boldsymbol{x = A\sin\theta}$

$$\therefore \sin\theta = \frac{x}{A} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{A}\right) \;\ldots\ldots (a)$$

$$dx = A\cos\theta\, d\theta \;\ldots\ldots (b)$$

$$A^2 - x^2 = A^2 - A^2\sin^2\theta = A^2(1-\sin^2\theta)$$

$$A^2 - x^2 = A^2\cos^2\theta$$

$$\sqrt{A^2-x^2} = A\cos\theta \;\ldots\ldots (c)$$

Sub. Eqns. (b) and (c) in Eq. (6)

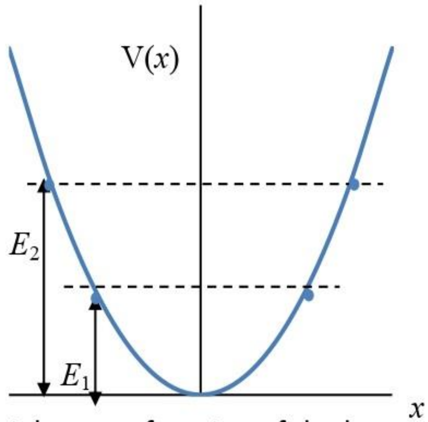
$$t = \sqrt{\frac{m}{k}} \int \frac{A\cos\theta}{A\cos\theta}\, d\theta$$

$$t = \sqrt{\frac{m}{k}}\theta + c \ldots\ldots (d)$$

Sub. Eq. (a) in Eq. (d)

$$t = \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{x}{A}\right) + c \ldots\ldots (13)$$

The value of x must lie between $\pm A\left(\pm\sqrt{\frac{2E}{k}}\right)$ in order for $\dot{x}$ to be real



Potential energy function of the harmonic oscillator.

1) at the upper point

$$x = X_{max}, v = 0$$

$$\therefore E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = 0 + \frac{1}{2}k$$

$$X_{max} = A$$

$$\therefore E = \frac{1}{2}kA^2 \rightarrow A = \sqrt{\frac{2E}{k}}$$

2) at the lower point

$$x = 0, v = v_{max}$$

$$\therefore E = \frac{1}{2}\, mv_{max}^2$$

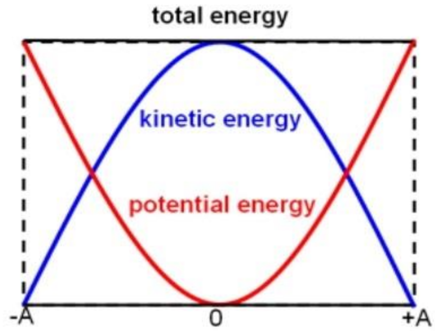
$$\frac{1}{2} m\, v_{max}^2 = \frac{1}{2} k A^2$$

$$v_{max}^2 = \frac{k\, A^2}{m}$$

$$v_{max} = \sqrt{\frac{k}{m}} A = A w_0$$ ***Maximum Velocity for Harmonic Oscillator***

$$A = \frac{v_{max}}{w_0}$$

As the particle oscillates, the kinetic and potential energies continually change. The constant total energy is entirely in the form of ***kinetic energy at the center***, where $x = 0$ and $\dot{x} = \pm v_{max}$ and it is all ***potential energy at extrema***, where $x = \pm A$ and $\dot{x} = 0$.



عندما يتأرجح الجسيم، تتغير الطاقة الكامنة والحركية باستمرار. تكون الطاقة الكلية ثابتة حيث تكون بشكل طاقة حركية في المركز و طاقة كامنة عند النهايات.