Subject Name: Numerical analysis

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Lecturer: Dr. Amir N.Saud

Email: amir-najah@uomus.edu.iq

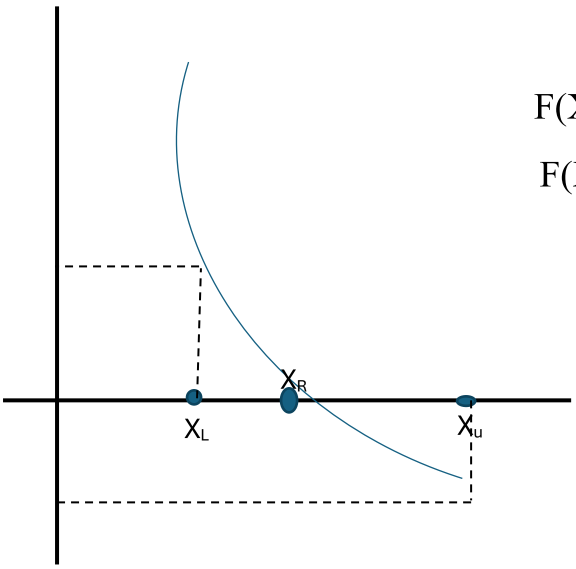
Lecture No. 2

Lecture Title: Methods for Finding the Roots - Approximate

First Solution of the non-linear equation

1- The Bisection Method

The bisection method or interval halving can be used to determine the solution to f(x) = 0 on an interval [x1 = a, x2 = b] if f(x) is real and continuous on the interval and $f(x_1)$ and $f(x_2)$ have opposite signs. We assume for simplicity that the root in this interval is unique. The location of the root is then calculated as lying at the midpoint of the subinterval within which the functions have opposite signs. The process is repeated to any specified accuracy .



$F(X_L)$= +ve

$F(X_u)$= -ve

$$the\ Root\ XR = \frac{Xl + Xu}{2}$$

The Root of the function when

1- $\varepsilon_a\% < \varepsilon_s\%$
2- $\varepsilon_a < \varepsilon_s$

Exp1/ Find the root of the function by the bisection method

$$F(x) = -0 \cdot 4\, x^2 + 2 \cdot 2\, x + 4 \cdot 7$$

Using X_L=5 , X_u=10 , ε_s %= 5%

Sol/

X_L	$F(x_L)$	X_u	$F(x_u)$	X_R	$F(x_r)$	$\varepsilon_a\%$
5	5.7	10	-13.3	7.5	-1.3	-

5	5.7	7.5	-1.3	6.25	2.825	20%
6.25	2.825	7.5	-1.3	6.87	0.917	9.09%
6.87	0.917	7.5	-1.3	7.185	-0.142	4.38%

Then the Root of the function when $\varepsilon_a\% < \varepsilon_s\% = 4.38\%$

Xr=7.185

Exp2/ Find the root of the function by bisection method

$$F(x) = x^2 - 3$$

Using $X_L=1$, $X_u=2$, using fifth iteration

Sol/

X_L	$F(x_L)$	X_u	$F(x_u)$	X_R	$F(x_r)$
1	-2	2	1	1.5	-0.75
1.5	-0.75	2	1	1.75	0.062
1.5	-0.75	1.75	0.062	1.625	-0.359
1.625	-0.359	1.75	0.062	1.687	-0.152
1.687	-0.152	1.75	0.062	1.718	-0.045

Exp 3/ Find the location of the smallest positive root of the equation $x^3 - 9x + 1 = 0$ and compute it by bisection method, correct to two decimal places.

Solution: To find the location of the **smallest positive** root
we tabulate the function $f(x) = x^3 - 9x + 1$ below:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-27	1	11	9	1	-7	-9	1	29

n	xo	x1	f(xo)	f(x1)	x2=(xo+x1)/2	f(x2)	update	new
1	0	1	1	-7	0.5	-3.375	x1=x2	0.5
2	0	0.5	1	-3.375	0.25	-1.234	x1=x2	0.25
3	0	0.25	1	-1.234	0.125	-1.123	x1=x2	0.125
4	0	0.125	1	-1.123	0.0625	0.4377	xo=x2	0.0625
5	0.0625	0.125	0.4377	-1.123	0.09375	0.15707397	xo=x2	0.09375
6	0.0938	0.125	0.15663	-0.12305	0.1094	0.01670934	xo=x2	0.1094
7	0.1094	0.125	0.01671	-0.12305	0.1172	-0.0531902	x1=x2	0.1172
8	0.1094	0.1175	0.01671	-0.05588	0.11345	-0.0195898	x1=x2	0.11345
9	0.1094	0.11345	0.01671	-0.01959	0.111425	-0.0014416	x1=x2	0.111425
10	0.1094	0.111425	0.01671	-0.00144	0.1104125	0.00763353	xo=x2	0.1104125
11	0.110413	0.1104125	0.00763	0.007634	0.11041275	0.00763129	xo=x2	0.11041275
12	0.1104128	0.1104128	0.00763	0.007631	0.11041275	0.00763129	xo=x2	0.11041275

From the above results, we conclude that the smallest root correct to two decimal places **is 0.11.**

#Homework

1- Find the Roof of the function

$$F(x) = x^2 - 3x$$

Using X_L=1 , X_u=2 , ε_s = 0.01

2- Find the Root of the function

$$F(x) = e^{-x}(3.2 \sin(x) - 0.5 \cos(x))$$

Using X_L=3 , X_u=4 , ε_s = 0.002

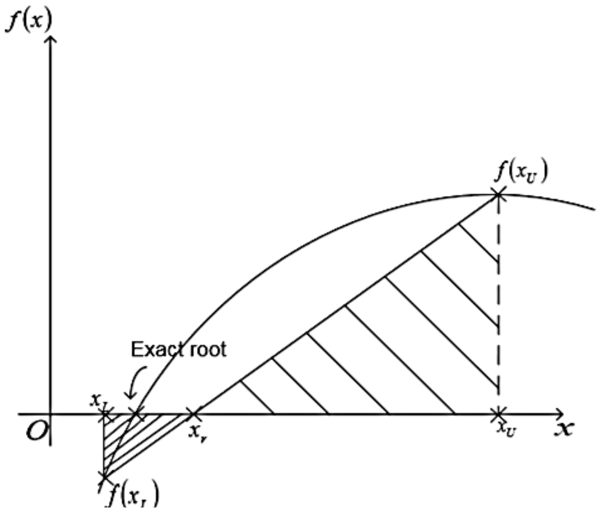
3- Find the root of function

$$F(x) = x^2 - 10$$

Using X_L=3 , X_u=4 , ε_s = 0.1

2- **False position methods**

In the previous method, the bisection method was described as one of the simple bracketing methods of solving a nonlinear equation of the general form f(x)=0



The above nonlinear equation can be stated as finding the value of x such that Equation (1) is satisfied.

In the bisection method, we identify proper values of X_l (lower bound value) and X_u (upper bound value)for the current bracket ,such that

$$f(Xl)\,f(Xu) < 0.$$

. (2)

The next predicted/improved root Xr, can be computed as the midpoint between XL and Xu as as

$$XR = \frac{Xl + Xu}{2}$$

The new upper and lower bounds are then established, and the procedure is repeated until the convergence is achieved (such that the new lower and upper bounds are sufficiently close to each other).

However, in the example shown in Figure 1, the bisection method may not be efficient because it does not take into consideration that $f(Xl)$ is much closer to the zero of the function $f(X)$ as compared to $f(Xu)$ .In other words, the next predicted root Xr would be closer to X_l (in the example as shown in Figure 1), than the mid-point between Xl and Xu. .

The false-position method takes advantage of this observation mathematically by drawing a Secant from the function value at Xl to the function value at Xu , and estimates the root as where it crosses the x-axis.

So to determine the Xr in false position method as shown

$$y - y1 = m(x - x1)$$

(x1,y1) at Xr = (Xr,0)

$$y - 0 = m(x - Xr)$$

$$m = \frac{y2 - y1}{x2 - x1}$$

So $$y = \frac{f(xl) - f(xu)}{Xl - Xu}\,(X - Xr)$$

By taking (x,y) any point so = (Xu,f(xu))

$$f(Xu) = \frac{f(xl) - f(xu)}{xl - xu}\,(Xu - Xr)$$

$$Xr = Xu - \frac{f(Xu) * (Xl - Xu)}{f(Xl) - f(Xu)}$$

$$x = x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Example 1/ Find the root of the function by false position method

$$F(x) = -0 \cdot 4\,x^2 + 2 \cdot 2\,x + 4 \cdot 7$$

Using X_L=5 , X_u=10 , ε_s %= 5%

Sol/

X_L	$F(x_L)$	X_u	$F(x_u)$	X_R	$F(x_r)$	ε_a%
5	5.7	10	-13.3	6.5	2.1	-
6.5	2.1	10	-13.3	6.9772	0.5770	6.84
6.9772	0.5770	10	-13.3	7.0987	0.1602	1.711

Then the Root of the function when $\varepsilon_a\% < \varepsilon_s\%$ =7.0987

Example 2/ Find the root of the function by the false position method

$$F(x) = (x-4)^2\,(x+2)$$

Using X_L=-2.5 , X_u=-1 , ε_s %= 0.1%

Sol /

X_L	$F(x_L)$	X_u	$F(x_u)$	X_R	$F(x_r)$	ε_a%
-2.5	-21.3	-1	25	-1.813	6.319	-
-2.5	-21.3	-1.813	6.319	-1.971	1.028	8.02
-2.5	-21.3	-1.971	1.028	-1.996	0.1542	1.229
-2.5	-21.3	-1.996	0.1542	-1.999	0.022	0.182
-2.5	-21.3	-1.999	0.022	-2	0.0033	0.027

Then, the Root of the function when ε_a% < ε_s% =-2

EXAMPLE 3: Determine the smallest positive root of $x - e^{-x} = 0$, correct of three significant figures using the Regula False method.

SOLUTION

$$\text{Here, } f(0) = 0 - e^{-0} = -1$$

$$\text{and } f(1) = 1 - e^{-1} = 0.63212.$$

X_L	$F(x_L)$	X_u	$F(x_u)$	X_R	$F(x_r)$
0	-1	1	0.63212	0.6127	0.0708
0.0708	0.0708	0.6127	0.0708	0.57219,0	
				0.5677	
				0.5672	
				0.5671	

3. Secant method

The secant method finds the root of a scalar-valued function f(x) of a single variable x when no information about the derivative exists. It is similar to the false position method but trades the possibility of non-convergence for faster convergence.

This method now requires two initial guesses, but unlike the bisection method, the two initial guesses do not need to bracket the root of the equation. The secant method is an open method and may or may not converge. However, when secant method converges, it will typically converge faster than the bisection method.

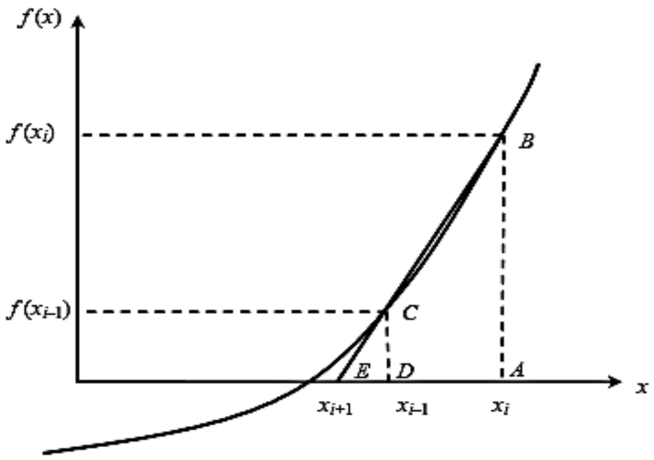
The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, x_{i-1}, X_i ,one draws a straight line between $f(X_i)$ and $f(X_{i-1})$ passing through the x-axis at x_{i+1} . ABE and DCE are similar triangles.

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$



Example 1/ Find the root of the function by *Secant method*

$$F(x) = e^{-x} - x$$

Using $X_{i-1}=0$, $X_0=1$, ε_s %= 1%

Sol/

X_{i-1}	$F(x_{i-1})$	X_i	$F(x_i)$	X_{i+1}	ε_a%
0	1	1	-0.63212	0.61270	63.2
1	-0.63212	0.61270	-0.0708	0.56384	8.67
0.61270	-0.0708	0.56384	0.00518	0.56717	0.58

Then the Root of the function when $\varepsilon_a\% < \varepsilon_s\%$ =0.56717

Example 2/ Find the root of the function by secant method

$$F(x) = x^3 - 0.165x^2 + 3.993 * 10^{-4}$$

Using X_{-1}=0.02 , X_0=0.05 , ε_s %= 0.06%

Sol/

X_{i-1}	$F(x_{i-1})$	X_i	$F(x_i)$	X_{i+1}	ε_a%
0.02	$3.41*10^{-4}$	0.05	$1.118*10^{-4}$	0.06461	22.62
0.05	$1.118*10^{-4}$	0.06461	$-1.981*10^{-5}$	0.06241	3.52
0.06461	$-1.981*10^{-5}$	0.06241	$-3.285*10^{-7}$	0.06238	0.059

Then the Root of the function when $\varepsilon_a\% < \varepsilon_s\%$ =0.06238

#Homework

1-find a root of the function $f(x) = cos(x) + 2\,sin(x) + x2$

Using X_{-1}=0 , X_0= -0.1 , ε_s %= 0.0005%