Subject Name: Numerical analysis

3rd Class, Second Semester

Academic Year: 2024-2025

Lecturer: Dr. Amir N.Saud

Email: amir-najah@uomus.edu.iq

Lecture No. 5

Lecture Title: Least square fitting: the best-fitted line; parabolic least square fitting)[Part 2)

2. System of linear equations

- Many engineering and scientific problems require the solution of a system of linear equations.
- We consider a system of m linear equations in n unknowns written as,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\ldots \quad \ldots \quad \ldots & \\
a_{m1} + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{m}x_n &= b_m
\end{aligned}
$$

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ b_m \end{bmatrix}
$$

- The solution of the non-homogeneous system exists, if the rank of the coefficient matrix A is equal to the rank of the augmented matrix [A : b] given by,

$$
[A : b] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} & b_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & an_3 & \ldots & a_{nn} & b_n \end{bmatrix}
$$

1. Classical Methods

Cramer's Rule: Let D = |A| be the determinant of the coefficient matrix A and D_i be the determinant obtained by replacing the i^{th} column of D by the column vector b. The Cramer's rule gives the solution vector x by the equations,

$$x_i = \frac{D_i}{D} \quad for\ i = 1,2,\ldots,n$$

Example: Use Cramer's rule to solve the following system

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = 2(-1-1) - 3(-1+3) + (-3-1) = -14$$

$$D_1 = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = (-1-1) - 3(-1+2) + (-2-1) = -8$$

$$D_2 = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 2(-2+1) + (-1+3) + (3-2) = 1$$

$$D_3 = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+2) - 3(2-3) + (-3-1) = 5$$

Hence by Cramer's rule, we get

$$x_1 = \frac{D_1}{D} = \frac{-8}{-14} = \frac{4}{7}, \quad x_2 = \frac{D_2}{D} = \frac{-1}{14}, \quad x_3 = \frac{D_3}{D} = -\frac{5}{14}$$

2. Elimination Methods

Matrix Inversion Method

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

Where Adj A is the adjoint matrix obtained by transposing the matrix of the cofactors of the elements a_{ij} of the determinant of the coefficient matrix A.

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{21} \cdots & A_{n1} \\ A_{12} & A_{22} \cdots & A_{n2} \\ \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} \cdots & A_{nn} \end{bmatrix}$$

A_{ij} being the cofactor of a_{ij}.

$$x = A^{-1}b$$

Example: Solve the given system of equations by matrix inversion method:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 13 \qquad \text{Adj } A = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}. \qquad A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -13 \\ 39 \\ 26 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Gaussian Elimination Method:

- This method consists in systematic elimination of the unknowns so as to reduce the coefficient matrix into an **upper triangular** system, which is then solved by the procedure of **back-substitution**.

Example: Solve the following system by Gauss elimination method:

$$\begin{aligned} x_1+2x_2+x_3&=0\\ 2x_1+2x_2+3x_3&=3\\ -x_1-3x_2&=2 \end{aligned} \quad \left[\begin{array}{ccccc} 1 & 2 & 1 & : & 0\\ 2 & 2 & 3 & : & 3\\ -1 & -3 & 0 & : & 2 \end{array}\right] \quad \begin{array}{c} \\ -2\\ 1 \end{array}\left[\begin{array}{ccccc} 1 & 2 & 1 & : & 0\\ 0 & -2 & 1 & : & 3\\ 0 & -1 & 1 & : & 2 \end{array}\right] \quad \begin{array}{c} \\ -1/2\\ \\ \end{array}\left[\begin{array}{ccccc} 1 & 2 & 1 & : & 0\\ 0 & -2 & 1 & : & 3\\ 0 & 0 & 1/2 & : & 1/2 \end{array}\right] \quad \boxed{x_1=1, x_2=-1, x_3=1}$$

Gauss-Jordan Elimination Method:

- The Gauss-Jordan elimination method is a variation of the Gaussian elimination method.
- In this method, the augmented coefficient matrix is transformed by row operations such that the coefficient matrix reduces to **the identity matrix**.

$$\left[\begin{array}{ccccc} 2 & 2 & 4 & : & 18\\ 1 & 3 & 2 & : & 13\\ 3 & 1 & 3 & : & 14 \end{array}\right] \left[\begin{array}{cccc} 1 & 2 & 4 & 18\\ 1 & 3 & 2 & 13\\ 3 & 1 & 3 & 14 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{ccccc} 1 & 1 & 2 & : & 9\\ 1 & 3 & 2 & : & 13\\ 3 & 1 & 3 & : & 14 \end{array}\right] \xrightarrow[R_3+2R_2]{R_2-R_1} \left[\begin{array}{ccccc} 1 & 1 & 2 & : & 9\\ 0 & 2 & 0 & : & 4\\ 0 & -2 & -3 & : & -13 \end{array}\right]$$

$$\left[\begin{array}{ccccc}1 & 1 & 2 & : & 9\\ 0 & 2 & 0 & : & 4\\ 0 & -2 & -3 & : & -13\end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{ccccc}1 & 1 & 2 & : & 9\\ 0 & 1 & 0 & : & 2\\ 0 & -2 & -3 & : & -13\end{array}\right] \xrightarrow[R_3+2R_2]{R_1-R_2} \left[\begin{array}{ccccc}1 & 0 & 2 & : & 7\\ 0 & 1 & 0 & : & 2\\ 0 & 0 & -3 & : & -9\end{array}\right]$$

$$\left[\begin{array}{ccccc}1 & 0 & 2 & : & 7\\ 0 & 1 & 0 & : & 2\\ 0 & 0 & -3 & : & -9\end{array}\right] \xrightarrow{R_3/(-3)} \left[\begin{array}{ccccc}1 & 0 & 2 & : & 7\\ 0 & 1 & 0 & : & 2\\ 0 & 0 & 1 & : & 3\end{array}\right] \xrightarrow{R_1-2R_3} \left[\begin{array}{ccccc}1 & 0 & 0 & : & 1\\ 0 & 1 & 0 & : & 2\\ 0 & 0 & 1 & : & 3\end{array}\right]$$

$$\boxed{x_1 = 1, x_2 = 2, x_3 = 3.}$$