Subject Name: Numerical analysis

3rd Class, Second Semester

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Lecture No. 6

Lecture Title: Least square fitting: the best-fitted line; parabolic least square fitting (part3)

3. Iterative Methods

We can use iteration methods to solve a system of linear equations when the coefficient matrix is diagonally dominant. This is ensured by the set of sufficient conditions given as follows,

$$\sum_{j=1,\ j\neq i}^{n} |a_{ij}| < |a_{ii}|, \quad \text{for } i = 1, 2, \ldots, n$$

Jacobi Iteration Method:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = b_3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = b_n$$

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n)/a_{11}$$
$$x_2 = (b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n)/a_{22}$$
$$x_3 = (b_3 - a_{31}x_1 - a_{32}x_2 - \ldots - a_{3n}x_n)/a_{33}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = (b_n - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{nn-1}x_{n-1})/a_{nn}$$

$$x_1^{(k+1)} = (b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_{1n}x_n^{(k)})/a_{11}$$
$$x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \ldots - a_{2n}x_n^{(k)})/a_{22}$$
$$x_3^{(k+1)} = (b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - \ldots - a_{3n}x_n^{(k)})/a_{33}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n^{(k+1)} = (b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \ldots - a_{nn-1}x_{n-1}^{(k)})/a_{nn}$$

The iterations are continued till the desired accuracy is achieved. This is checked by the relations,

$$\left|x_i^{(k+1)} - x_i^{(k)}\right| < \varepsilon, \quad \text{for } i = 1, 2, \ldots, n$$

Gauss-Seidel Iteration Method:

This is a simple modification of the Jacobi iteration. In this method, at any stage of iteration of the system, the improved values of the unknowns are used for computing the components of the unknown vector.

$$x_1^{(k+1)} = (b_1 - a_{12} x_2^{(k)} - a_{13} x_3^{(k)} - \ldots - a_{1n} x_n^{(k)}) / a_{11}$$
$$x_2^{(k+1)} = (b_2 - a_{21} x_1^{(k+1)} - a_{23} x_3^{(k)} - \ldots - a_{2n} x_n^{(k)}) / a_{22}$$
$$x_3^{(k+1)} = (b_3 - a_{31} x_1^{(k+1)} - a_{32} x_2^{(k+1)} - \ldots - a_{3n} x_n^{(k)}) / a_{33}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n^{(k+1)} = (b_n - a_{n1} x_1^{(k+1)} - a_{n2}^{(k+1)} x_2 (k+1) - \ldots - a_{nn-1} x_{n-1}^{(k+1)}) / a_{nn}$$

- It is clear from above that for computing $x_2^{(k+1)}$, the improved value of $x_1^{(k+1)}$ is used instead of; $x_1^{(k)}$ and for computing $x_3^{(k+1)}$ the improved values $x_1^{(k+1)}$ and $x_2^{(k+1)}$ are used.
- Finally, for computing $x_n^{(k)}$, improved values of all the components $x_1^{(k+1)}$, $x_2^{(k+1)}$,..., $x_{n-1}^{(k+1)}$ are used.
- Further, as in the Jacobi iteration, the iterations are continued till the desired accuracy is achieved.

Example: Solve the following system by Gauss-Seidel iteration method.

$$20x_1 + 2x_2 + x_3 = 30$$
$$x_1 - 40x_2 + 3x_3 = -75$$
$$2x_1 - x_2 + 10x_2 = 30$$

It is evident that the coefficient matrix is diagonally dominant and the sufficient conditions for convergence of the Gauss-Seidel iterations are satisfied, since:

$$|a_{11}| = 20 \geq |a_{12}| + |a_{13}| = 3$$
$$|a_{22}| = 40 \geq |a_{21}| + |a_{23}| = 4$$
$$|a_{33}| = 10 \geq |a_{31}| + |a_{32}| = 3$$

For starting the iterations, we rewrite the equations as,

$$x_1 = \frac{1}{20}(30 - 2x_2 - x_3)$$
$$x_2 = \frac{1}{40}(75 + x_1 + 3x_3)$$
$$x_3 = \frac{1}{10}(30 - 2x_1 + x_2)$$

The initial approximate solution is taken as,

$$x_1^{(0)} = 1.5, \quad x_2^{(0)} = 2.0, \quad x_3^{(0)} = 3.0$$

The first iteration gives,

$$x_1^{(1)} = \frac{1}{20}(30 - 2\times 2.0 - 3.0) = 1.15$$
$$x_2^{(1)} = \frac{1}{40}(75 + 1.15 + 3\times 3.0) = 2.14$$
$$x_3^{(1)} = \frac{1}{10}(30 - 2\times 1.15 + 2.14) = 2.98$$

The second iteration gives,

$$x_1^{(2)} = \frac{1}{20}(30 - 2\times 2.14 - 2.98) = 1.137$$
$$x_2^{(2)} = \frac{1}{40}(75 + 1.137 + 3\times 2.98) = 2.127$$
$$x_3^{(2)} = \frac{1}{10}(30 - 2\times 1.137 + 2.127) = 2.986$$

The third iteration gives,

$$x_1^{(3)} = \frac{1}{20}(30 - 2\times 2.127 - 2.986) = 1.138$$
$$x_2^{(3)} = \frac{1}{40}(75 + 1.138 + 3\times 2.986) = 2.127$$
$$x_3^{(3)} = \frac{1}{10}(30 - 2\times 1.138 + 2.127) = 2.985$$

Thus the solution correct to three significant digits can be written as x1 = 1.14, x2 = 2.13, x3 = 2.98.

Example: Solve the following system by Gauss-Seidel iterative method correct upto four significant digits.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Solution: The given system is clearly having diagonally dominant coefficient matrix,

i.e.,
$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n$$

$$x_1^{(k+1)} = (b_1 - a_{12}\, x_2^{(k)} - a_{13} x_3^{(k)} - \ldots - a_{1n}\, x_n^{(k)}) / a_{11}$$
$$x_2^{(k+1)} = (b_2 - a_{21}\, x_1^{(k+1)} - a_{23} x_3^{(k)} - \ldots - a_{2n}\, x_n^{(k)}) / a_{22}$$
$$x_3^{(k+1)} = (b_3 - a_{31}\, x_1^{(k+1)} - a_{32} x_2^{(k+1)} - \ldots - a_{3n}\, x_n^{(k)}) / a_{33}$$

....

$$x_n^{(k+1)} = (b_n - a_{n1}\, x_1^{(k+1)} - a_{n2}^{(k+1)} x_2(k+1) - \ldots - a_{nn-1}\, x_{n-1}^{(k+1)}) / a_{nn}$$

$$x_1^{(k+1)} = 0.3 + 0.2 x_2^{(k)} + 0.1\, x_3^{(k)} + 0.1\, x_4^{(k)}$$
$$x_2^{(k+1)} = 1.5 + 0.2 x_1^{(k+1)} + 0.1\, x_3^{(k)} + 0.1\, x_4^{(k)}$$
$$x_3^{(k+1)} = 2.7 + 0.1 x_1^{(k+1)} + 0.1\, x_2^{(k+1)} + 0.2\, x_4^{(k)}$$
$$x_4^{(k+1)} = -0.9 + 0.1 x_1^{(k+1)} + 0.1\, x_2^{(k+1)} + 0.2\, x_3^{(k+1)}$$

We start the iteration with,

$$x_1^{(0)} = 0.3,\ x_2^{(0)} = 1.5,\ x_3^{(0)} = 2.7,\ x_4^{(0)} = -0.9$$

The results of successive iterations are given in the table below.

k	x_1	x_2	x_3	x_4
1	0.72	1.824	2.774	–0.0196
2	0.9403	1.9635	2.9864	–0.0125
3	0.09901	1.9954	2.9960	–0.0023
4	0.9984	1.9990	2.9993	–0.0004
5	0.9997	1.9998	2.9998	–0.0003
6	0.9998	1.9998	2.9998	–0.0003
7	1.0000	2.0000	3.0000	0.0000

Hence the solution correct to four significant figures is:
$x_1 = 1.0000$, $x_2 = 2.000$, $x_3 = 3.000$, $x_4 = 0.000$.

Example: Find the inverse of the following matrix A by Gaussian elimination method.

$$A=\begin{bmatrix}2 & 3 & -1\\ 4 & 4 & -3\\ 2 & -3 & 1\end{bmatrix} \qquad [A.I]=\begin{bmatrix}2 & 3 & -1 & : & 1 & 0 & 0\\ 4 & 4 & -3 & : & 0 & 1 & 0\\ 2 & -3 & 1 & : & 0 & 0 & 1\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix}2 & 3 & -1 & : & 1 & 0 & 0\\ 0 & -2 & -1 & : & -2 & 1 & 0\\ 0 & -6 & 2 & : & -1 & 0 & 1\end{bmatrix}$$

$$\xrightarrow[R_3-3R_2]{} \begin{bmatrix}2 & 3 & -1 & : & 1 & 0 & 0\\ 0 & -2 & -1 & : & -2 & 1 & 0\\ 0 & 0 & 5 & : & 5 & -3 & 1\end{bmatrix} \qquad \begin{bmatrix}2 & 3 & -1 & : & 1\\ 0 & -2 & -1 & : & -2\\ 0 & 0 & 5 & : & 5\end{bmatrix}\begin{bmatrix}2 & 3 & -1 & : & 0\\ 0 & -2 & -1 & : & 1\\ 0 & 0 & 5 & : & -3\end{bmatrix}\begin{bmatrix}2 & 3 & -1 & : & 0\\ 0 & -2 & -1 & : & 0\\ 0 & 0 & 5 & : & 1\end{bmatrix}$$

The solution of the three are easily derived by back-substitution, which give the three columns of the inverse matrix given below:

$$\begin{bmatrix}1/4 & 0 & 1/4\\ 1/2 & -1/5 & -1/10\\ 1 & -3/5 & 1/5\end{bmatrix}$$

Example: Compute the inverse of the following matrix by Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix} \qquad [A:I] = \left[\begin{array}{ccc:ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1/2]{} \left[\begin{array}{ccc:ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-4R_1]{R_3-2R_1} \left[\begin{array}{ccc:ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2/-2]{} \left[\begin{array}{ccc:ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & +1/2 & 1 & -1/2 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+6R_2]{R_1-3R_2/2} \left[\begin{array}{ccc:ccc} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 5 & 5 & -3 & 1 \end{array}\right] \xrightarrow[R_3/5]{} \left[\begin{array}{ccc:ccc} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$$

$$\xrightarrow[R_2-1R_3/2]{R_1+5R_3/4} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 & -1/5 & -1/10 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$$

Which gives $A^{-1} = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 1/2 & -1/5 & -1/10 \\ 1 & -3/5 & 1/5 \end{bmatrix}$