Root Locus

Basic concepts of root locus

In the previous sections, we have studies that the stability of a system. It depends on the location of the roots of the characteristic equation. We can also say that the stability of the system depends on the location of closed-loop poles. Such knowledge of the movement of the poles in the s-plane when the parameters are varied is important. The minor changes in the parameters can greatly help in the system designing. The nature of the system's transient response is closely related to the location of the poles in the s-plane. We have also studied the Routh Hurwitz criteria that describe the stability of the algebraic equation. If any of the term in the first column of the Roth table possesses a sign change, the system tends to become unstable.

The root locus method was introduced by W.R Evans in 1948. Root locus is a graphical method in which the movement of poles in the s-plane can be located when a specific parameter is varied from 0 to infinity. The parameter assumed to be varied is generally the gain of the system.

The equation of a closed loop system is given by:

$1 + G(s)H(s) = 0$

Where: G(s) is the gain of the transfer function H(s) is the feedback gain

In the case of root locus, the gain K is also assumed as part of the closed-loop system. K is known as system gain or the gain in the forward path. The characteristic equation after including the forward gain can be represented as:

$1 + KG'(s)H(s) = 0$

Where: $G(s) = KG'(s)$

When the system includes the variable parameter K, the roots of the closed loop system are now dependent on the values of 'K.' The value of 'K' variable can vary in two cases, as shown below:

1. $-\infty$ to $+\infty$
2. 0 to $+\infty$

In the first case, for every different value (integer or decimal) of K, we will get separate set of locations of the roots. If all such locations are joined, the resulting plot is defined as the root locus. We can also define root locus as the locus of the closed loop poles obtained when the system gain 'K' is varied from -infinity to infinity.

When the K varies from zero to infinity, the plot is called the direct root locus. If the system gain 'K' varies from -infinity to zero, the plot thus obtained is known as inverse root locus. The gain K is generally assumed from zero to infinity unless specially stated.

Root Locus Construction Rules

1. Starting points (K = 0). The root loci start at the open-loop poles.
2. Termination points (K = ∞). The root loci terminate at the open-loop zeros when they exist, otherwise at ∞
3. Number of distinct root loci (branches): This is equal to the order of the characteristic equations (or the number of poles of open loop transfer function).
4. Symmetry of root loci: The root loci are symmetric about the real axis.
5. Root locus locations on the real axis: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
6. Break away (in) points. The points at which a locus breaks away from (or break in) the real axis can be found by letting K as a function of s , taking the derivative of (dK /ds) and then setting the derivative equal to zero.
7. RHS, crossover: This can be obtained by determining the value of K for marginal stability Routh-Hurwitz criterion.

Rules for Constructing a Root Locus

Rule 1

The root locus is symmetric with respect to the real axis.

Rule 2

The root locus originates at the poles of $G(s)H(s)$ (for $K=0$) and terminates at the zeros of $G(s)H(s)$ (as $K \to \infty$), including zeros at infinity.

Rule 3

If the open-loop function has α zeros at infinity, the root locus approaches α asymptotes as $K \to \infty$. The asymptotes are located at angles

$$\theta = \frac{r180^\circ}{\alpha}, \qquad r = \pm 1, \pm 3, \pm 5, \ldots$$

and intersect the real axis at the point

$$\sigma_a = \frac{\sum(\text{poles}) - \sum(\text{finite zeros})}{\alpha}, \qquad (\alpha \geq 2 \text{ only})$$

Here, $\alpha = n - m$ (zeros at infinity)

where n = number of poles
m = number of finite zeros

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Rule 4

The root locus includes all points on the real axis to the left of an odd number of poles and/or finite zeros.

Rule 5

Breakaway points are given by the roots of

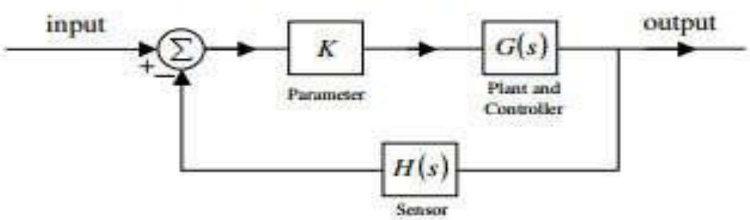
$$\frac{d[G(s)H(s)]}{ds}=0$$

or, equivalently,

$$N(s)D'(s)-D(s)N'(s)=0.$$

Here, $N(s)$ and $D(s)$ are the numerator and denominator polynomials of $G(s)H(s)$.

General Procedure



1. Obtain the open-loop function $KG(s)H(s)$.

2. Quantify the number of poles and finite zeros (n and m) of $G(s)H(s)$ and their locations. Plot the poles and finite zeros in the complex s-plane. (Denote poles and zeros by the symbols $\times$ and o, respectively.)

3. Quantify the number of zeros at infinity (same as the number of asymptotes) using $\alpha=n-m$.

4. Per rule 4, the root locus includes all points on the real axis to the left of an odd number of poles and finite zeros. Include these points in the root locus.

5. If $\alpha\geq 2$, quantify the angles of the α asymptote(s) using

$$\theta=\frac{r180^{\circ}}{\alpha},\qquad r=\pm1,\pm3,\pm5,\ldots$$

Find the point (if it exists) at which the asymptote(s) intersect(s) the real axis using

$$\sigma_a=\frac{\sum(\text{poles})-\sum(\text{finite zeros})}{\alpha}.$$

Sketch the asymptote(s).

6. Quantify, as appropriate, the breakaway points by calculating the roots of

$$\frac{d[G(s)H(s)]}{ds}=0$$

or $$N(s)D'(s)-D(s)N'(s)=0.$$

7. Finish sketching the root locus.

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Examples:

Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and −1 and between −2 and −3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

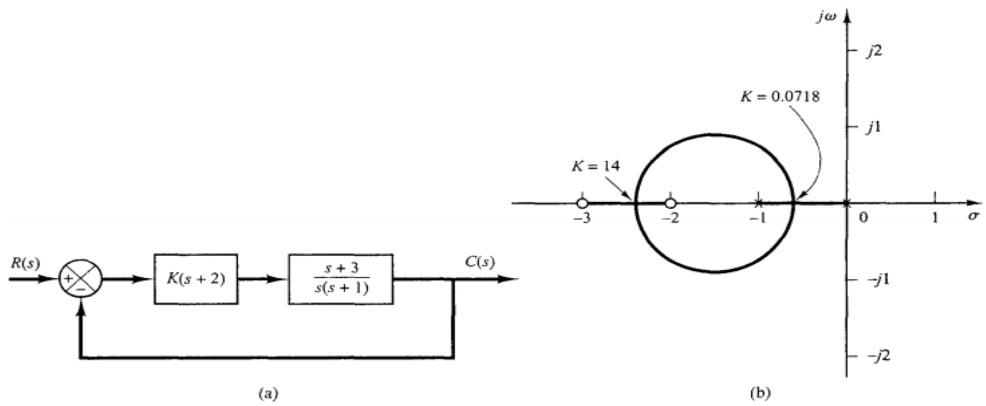

Figure 6–39
(a) Control system; (b) root-locus plot.

3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s+2)(s+3)}{s(s+1)} = 0$$

or

$$K = -\frac{s(s+1)}{(s+2)(s+3)}$$

The breakaway and break-in points are determined from

$$\frac{dK}{ds} = -\frac{(2s+1)(s+2)(s+3) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2}$$
$$= -\frac{4(s+0.634)(s+2.366)}{[(s+2)(s+3)]^2}$$
$$= 0$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

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$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at -1.5 that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–39(b).

Note that this system is stable for any positive value of K since all the root loci lie in the left-half s plane.

Small values of K ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of K ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of K ($14 < K$) correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K.

The value of K should be adjusted so that system performance is optimum according to a given performance index.

Example:

Sketch the root loci of the control system shown in Figure 6–40(a).

Solution. The open-loop poles are located at $s = 0$, $s = -3 + j4$, and $s = -3 - j4$. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, 180°$$

Referring to Equation (6–13), the intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{0 + 3 + 3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

$$K = -s(s^2 + 6s + 25)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 12s + 25) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$

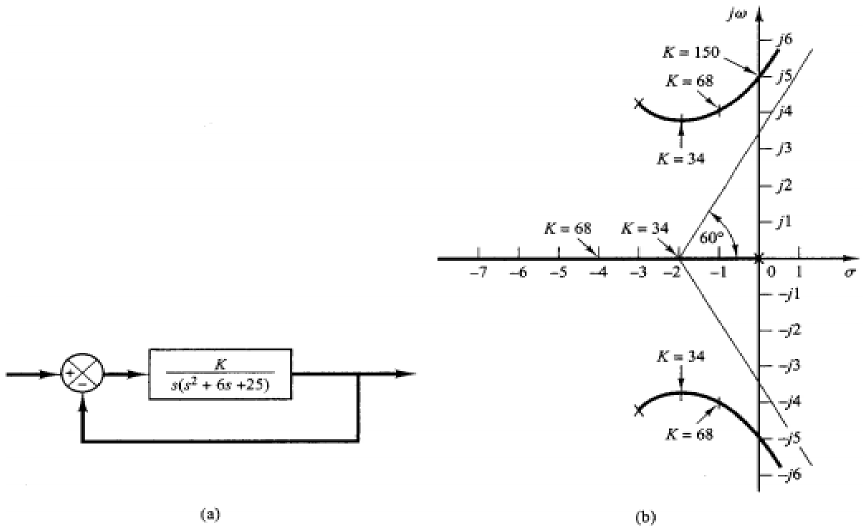

Figure 6–40
(a) Control system; (b) root-locus plot.

Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of K, we obtain

$$K = -s(s^2 + 6s + 25)\Big|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of K must be real and positive.)

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for ω and K as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \quad K = 150 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain K at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain K is increased, then other closed-loop poles must move toward the left as gain K is increased. This fact can be seen clearly in this problem. If the gain K is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.