2.1Graphical solution-Mohr's stress circle and Triaxial States of Stress:

The Graphical method called Mohr's stress circle makes the solution very easy and interesting and minimizes the probability of errors. Mohr's s tress circle is constructed based on the transformation equations for plane stress. Recall the transformation equations:

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)cos2\theta - \tau_{xy}sin2\theta$$

$$\tau_{\theta_s} = \frac{1}{2}(\sigma_x - \sigma_y)sin2\theta_s + \tau_{xy}cos2\theta_s$$

By squaring both equations, adding the results.

σ_x, σ_y and τ_{xy} are known constants defining the specified state of stress, whereas σ and τ are variables. Thus, the above equation can be written as follow:

$$(\sigma - C)^2 + \tau^2 = R^2$$

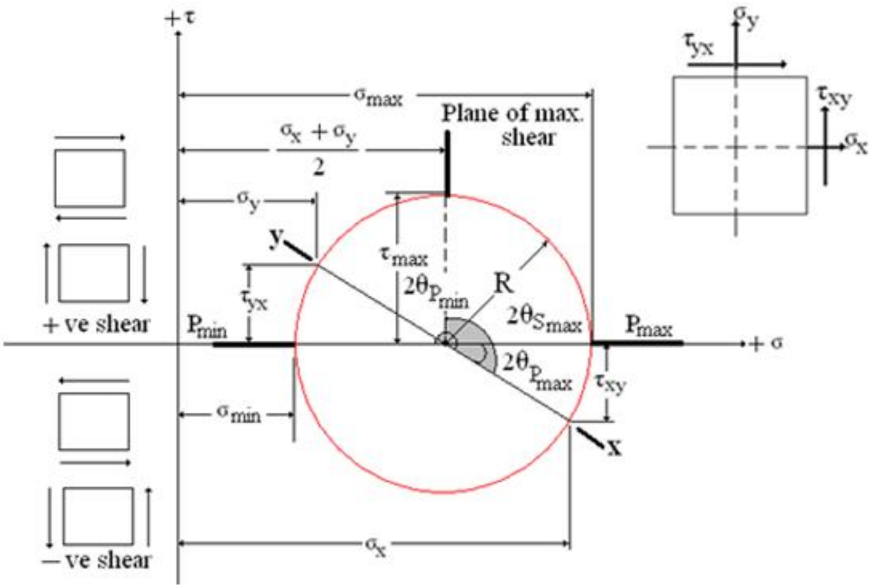
This is readily recognized as a circle with σ and τ representing its coordinates and has a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The center of the circle is off-set rightward a distance:

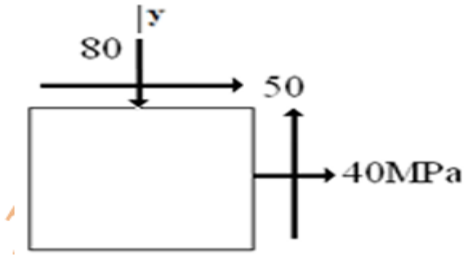
$$c = \frac{\sigma_x + \sigma_y}{2}$$

This circle was first introduced by the German engineer Otto Mohr (1835–1918) and is known as Mohr's circle for plane stress.



Example#3:

Construction of Mohr's circle for the data of Example (1):

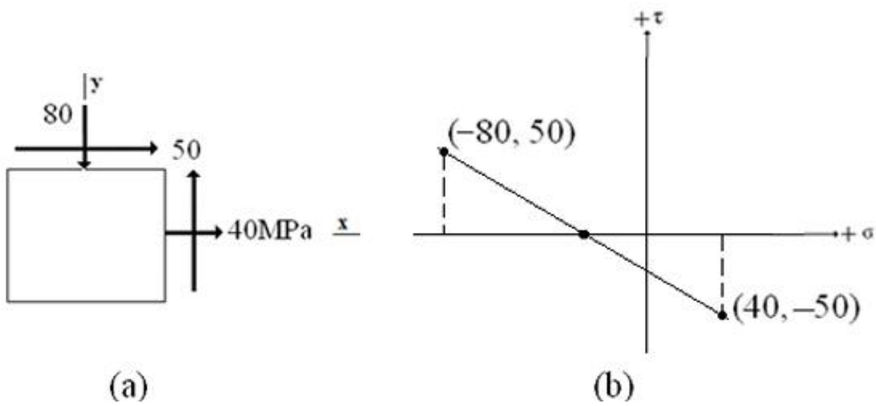


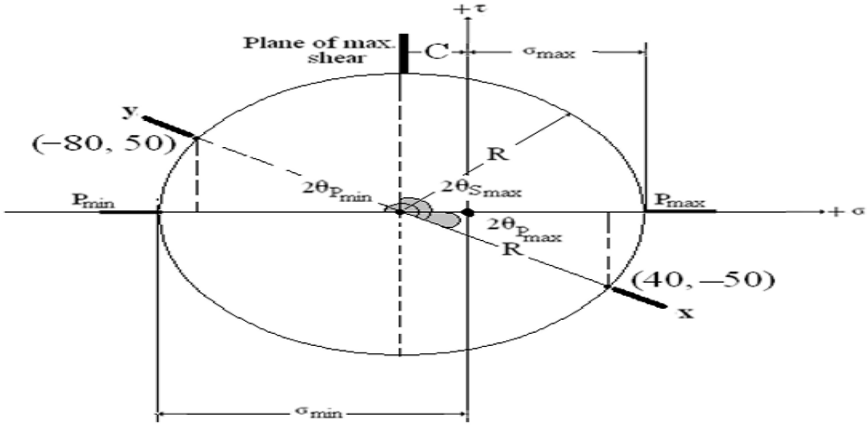
Solution:-

$$C = \frac{40+(-80)}{2} = -20MPa$$

$$tan2\theta_p = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{-2*50}{40-(-80)}$$

$$2\theta p = +39.8^0 \; , \; \theta p = +19.9^0$$





$$R = \sqrt{\left(\frac{\sigma_{x-}\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

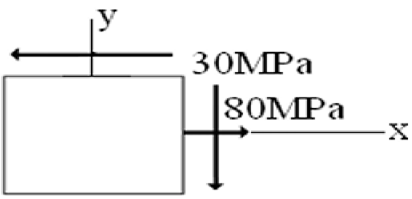
$$\tau \text{ max} = R = \sqrt{(\frac{40-(-80)}{2})^2 + +50^2} = 78.1\text{Mpa}$$

σmax =R-C=78.1-20= 58.1 MPa(Tension)

σ_{min} = R+c=78.1+20=98.1MPa(Compression)

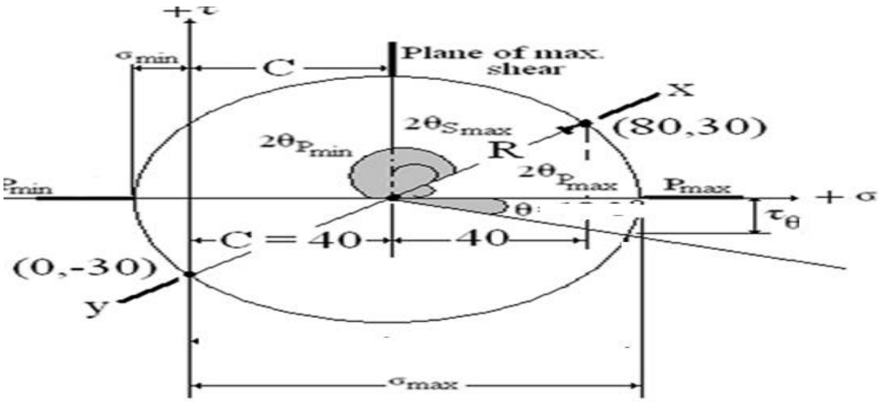
$$2\theta_{s_{max}} = 39.8 + 90 \quad \rightarrow \theta_{s_{max}} = 64.9^0$$

Example#4: For the stress element shown, determine the two principal stresses, the maximum shear stress, and the planes at which these stresses act. Draw your results on elements oriented properly with respect to x and y directions.



Solution:-

$$C = \frac{80+0}{2} = 40MPa$$



$$tan2\theta_p = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\rightarrow 2\theta p = -36.8^0$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = R = \sqrt{(\frac{80-(0)}{2})^2 + +30^2} = 50 \text{ Mpa}$$

σ_{max} =R+C=50+40= 90 MPa(Tension)

σ_{min} = R-c=50-40=10MPa(Compression)

$$2\theta_{s_{max}} = 90 - 36.8 = 53.2 \quad \rightarrow \theta_{s_{max}} = 26.6^0$$

2.2 Elastic Stress-Strain Relations (for isotropic solids):

stress	Strain in the x-direction	Strain in the y-direction	Strain in the z-direction
σ_{XX}	$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$	$\varepsilon_{yy} = -\frac{\upsilon\sigma_{xx}}{E}$	$\varepsilon_{zz} = -\frac{\upsilon\sigma_{xx}}{E}$
σ_{yy}	$\varepsilon_{xx} = -\frac{\upsilon\sigma_{yy}}{E}$	$\varepsilon_{YY} = \frac{\sigma_{YY}}{E}$	$\varepsilon_{ZZ} = -\frac{\upsilon\sigma_{YY}}{E}$
σ_{ZZ}	$\varepsilon_{xx} = -\frac{\upsilon\sigma_{zz}}{E}$	$\varepsilon_{YY} = -\frac{\upsilon\sigma_{zz}}{E}$	$\varepsilon_{ZZ} = \frac{\sigma_{ZZ}}{E}$

$$\varepsilon_{XX} = \frac{1}{E}[\sigma_{XX} - \upsilon(\sigma_{YY} + \sigma_{ZZ})]$$

$$\varepsilon_{YY} = \frac{1}{E}[\sigma_{YY} - \upsilon(\sigma_{XX} + \sigma_{ZZ})]$$

$$\varepsilon_{ZZ} = \frac{1}{E}[\sigma_{ZZ} - \upsilon(\sigma_{XX} + \sigma_{YY})]$$

$$\tau_{XY} = G\gamma_{XY}; \quad \tau_{YZ} = G\gamma_{YZ}; \quad \tau_{XZ} = G\gamma_{XZ}; \quad G = \frac{E}{2(1+\upsilon)}; \quad K = \frac{E}{3(1-2\upsilon)}$$

K- Bulk modulus (the ratio of stress to strain under uniform pressure conditions = (volumetric stress)/(volumetric strain)) and ν=Poisson ratio

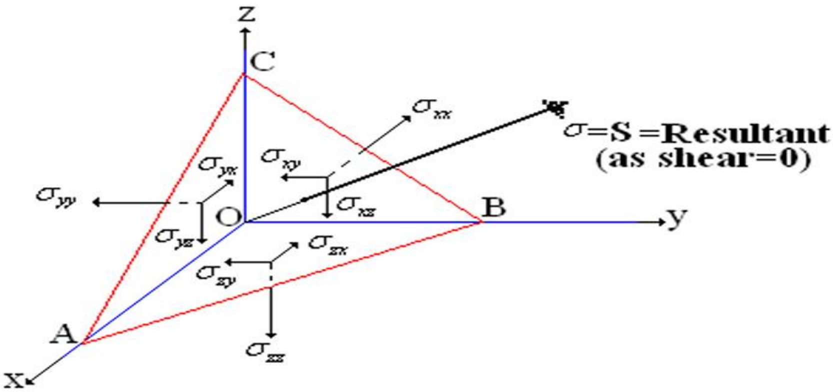
2.3 Triaxial States of Stress

The state of stress at a point can be defined in terms of six stresses:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ . & \sigma_{yy} & \sigma_{yz} \\ . & . & \sigma_{zz} \end{bmatrix} \text{ where } \begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix}$$

In general, a three-dimensional state of stress consists of several unequal stresses acting at a point. When the stress components are principal, we refer to the state of stress as triaxial. The state of stress is known as cylindrical if two of the principal stresses are equal (e.g., $\sigma_1 = \sigma_2 \neq \sigma_3$).

The state of stress is known as **hydrostatic** if all three principal stresses are equal (e.g., $\sigma_1 = \sigma_2 = \sigma_3$). Let ABC represent a principal plane (of area A) cutting through a unit cube. Let σ represent the resultant principal stress acting on the principal plane.



Let l, m, n, respectively, equal the direction cosines corresponding to the angles between the stress σ and the x-,y-, and z-axes. As noted above, in this problem we are assuming that the stress is perpendicular to the plane ABC. On the inclined plane, the stress σ will have components that are normal to the plane (and shear components that are parallel to the plane, There is no shear on the plane

$$\begin{bmatrix} \sigma-\sigma_{xx} & -\sigma_{yx} & -\sigma_{zx} \\ -\sigma_{xy} & \sigma-\sigma_{yy} & -\sigma_{zy} \\ -\sigma_{xz} & -\sigma_{yz} & \sigma-\sigma_{zz} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

The solution of the determinant of the matrix on the left yields a cubic equation in terms of σ:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \qquad I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \qquad \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

The quantities I_1, I_2, $I_3$are known as **stress invariants** because for a given stress state, they are the same in all coordinate systems.

Example 1: The state of stresses at a point is given by: Calculate the **stress invariants**

$$\begin{bmatrix} 500 & 500 & -400 \\ & -300 & 300 \\ & & -1000 \end{bmatrix}$$

Solution:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 500 - 300 - 1000 = -800MPa$$

$$\begin{aligned} I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}{}^2 + \sigma_{yz}{}^2 - \sigma_{zx}{}^2 = \\ = 500(-300) + (-300)(-1000) \\ + (-1000)(500) - (500)2 - (300)2 - (-400)2 \\ = -850\text{GPa} \end{aligned}$$

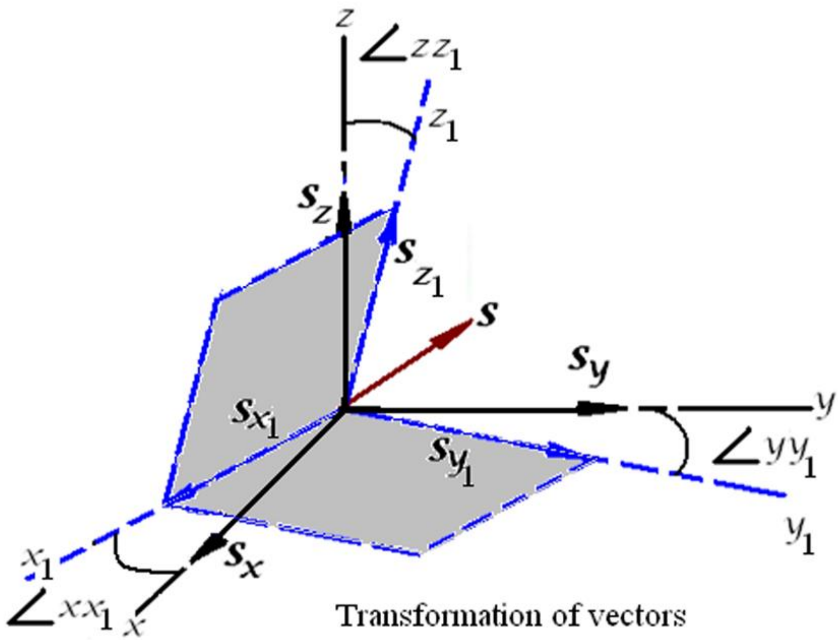
$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}{\sigma_{yz}}^2 + \sigma_{yy}{\sigma_{zx}}^2 - \sigma_{zz}{\sigma_{xy}}^2$

$$= 500(-300)(-1000) + 2(500)(300)(-400) - (500)(300)2 - (-300)(400)2 - (-1000)(500)2 = 23810^6 MPa$$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$ So $\sigma^3 + 800\sigma^2 - 85 * 10^4\sigma - 283 * 10^6 = 0$

2.4 Transformation of Coordinates:

Nearly all solid objects have non-uniform structures (microscopically and macroscopically). Thus the states of stress generally vary from point to point even though the applied forces do not change. Sometimes it is necessary and/or convenient to express the stresses and strains relative to different coordinate systems. A convenient method is needed to “transform” stresses (strains, elastic properties) from one coordinate system to another. The vector S can be easily resolved into components that are parallel to any set of reference axes. The two coordinate systems are related through a series of angles. We define them in terms of direction cosines.



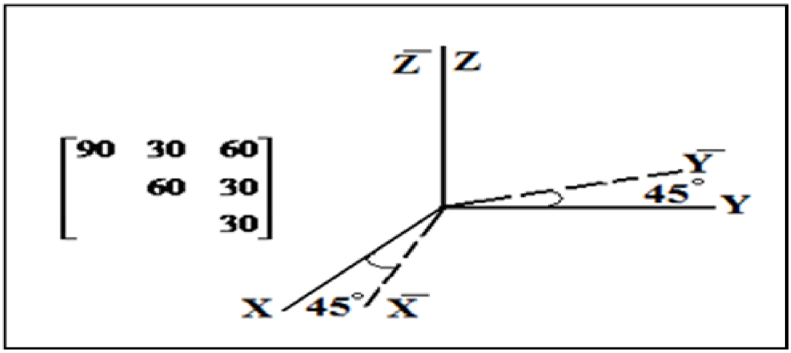
Transformation of vectors

Coordinates		S x	S y	S z
S_k	x_1	cos<xx_1	cos<yx_1	cos<zx_1
	y_1	cos<xy_1	cos<yy_1	cos<zy_1
	z_1	cos<xz_1	cos<yz_1	cos<zz_1

OR

Coordinates		S x	S y	S z
S_k	x_1	l_x	l_y	l_z
	y_1	m_x	m_y	m_z
	z_1	N_x	n_y	n_z

Example: Transform the stress state below to the new axes shown in the fig.



Solution

	1 X	2 Y	3 Z
1 X'	45^0	45^0	90^0
2 Y'	135^0	45^0	90^0
3 Z'	90^0	90^0	0^0

Coordinates	**S**		
	x	**y**	**z**
X/	**lx=0.7**	**ly=0.7**	**lz=o**
y/	**mx=-0.7**	**my=0.7**	**mz=o**
z/	**nx=1**	**ny=1**	**nz=1**

2.5 Strains

An infinitesimal normal strain is defined by the change of length, L, of a line:

$d\varepsilon = dL/L$ Integrating from the initial length, L_0, to the current length, L,

$\varepsilon = \ln(L/L_0)$.

This finite form is called **true strain**(or **natural strain, logarithmic strain**). Alternatively, **engineering or nominal strain, e,** is defined as

$e = \Delta L/L_0$.

Example 1: An element 1 cm long is extended to twice its initial length (2 cm) and then compressed to its initial length (1 cm). 1. Find true strains for the extension and compression. 2. Find engineering strains for the extension and compression.

SOLUTION:

1. During the extension, $\varepsilon = \ln(L/L_0) = \ln 2 = 0.693$, and during the compression, $\varepsilon = \ln(L/L_0) = \ln(1/2) = -0.693$.
2. During the extension, $e = \Delta L/L_0 = 1/1 = 1.0$, and during the compression, $e = \Delta L/Lo = -1/2 = -0.5$.