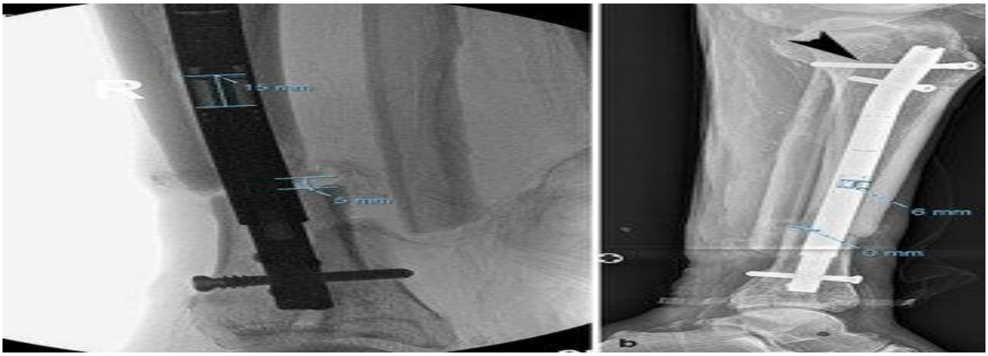
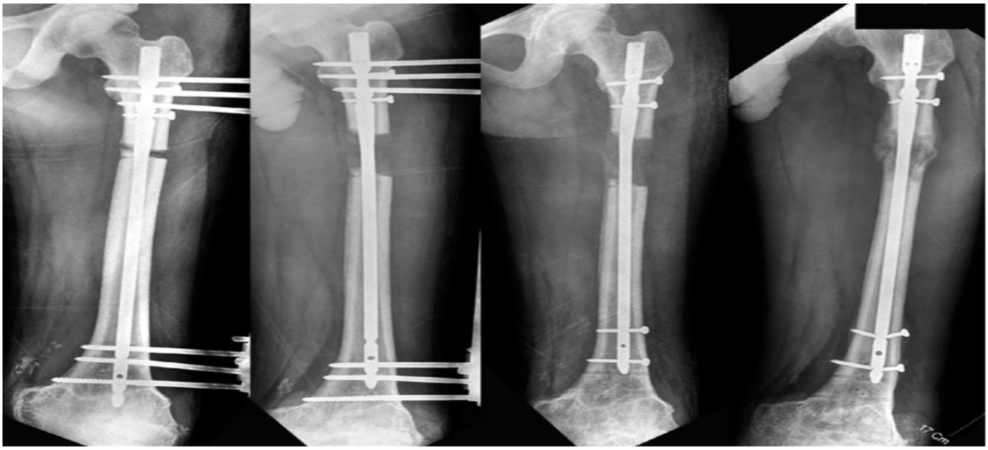
Deflections of Bones (Beam)

1. Introduction:

The transverse loads applied to a beam cause it to deflect in a direction perpendicular to its longitudinal axis. This initially straight axis is deformed into a curve, called the deflection curve or the elastic curve of the beam, which characterize its shape after the deformation. Determining the equation of the elastic curve enable us to determine the magnitude and location of maximum deflection of the beam under a given loading, since the design specifications of any beam is generally include a maximum permissible value for its deflection. Of interest is that the deflections at specific points along the axis of the beam can be calculated.

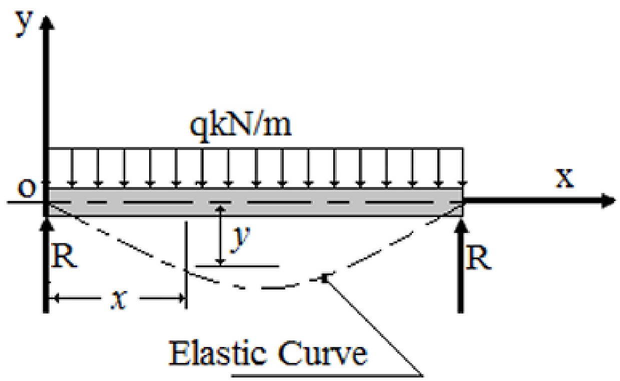
Accurate measurement of strains and deformation is an essential part of the analysis of the beams, particularly in the biomechanics. The assessment of the behaviour of the material and its structural integrity under different load scenarios requires an understanding of these deformations. This is particularly important when designing medical support structures such as orthotic devices to stabilize and maintain the alignment of the two ends of a severely broken bone during its healing. Such devices shall be designed to withstand excessive deformation under normal biomechanical loads encountered during daily activities in order to be effective. More generally, many biomechanical applications, from orthopaedic implant design to prosthetic design, where ensuring structural stability and functional efficiency are essential, depend on the ability to measure, predict and control the curvature of the beam.





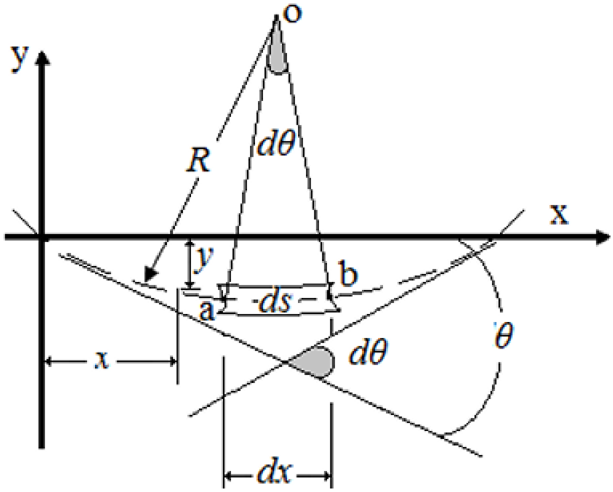
2. Elastic curve equation:

To obtain a general equation for the elastic curve, the simply supported beam loaded with a consider distributed load, as shown in the figure. We uniformly of coordinates at the left end of the take the origin directed to the right and y-axis beam, with the x-axis of the beam. At any point at representing the deflection displacement of the distance "x" from the origin the the beam at that point in y-direction is the deflection of point.



Simply supported beam loaded with a uniformly distributed load

Consider two points "a&b" located on the elastic curve with a separating distance between them equals "ds" along the curve as shown.



Elastic curve

For infinitesimally small "dx", we can write:

$$\tan\theta = \frac{dy}{dx} \text{ or } \theta = \frac{dy}{dx} \text{ and } ds \equiv dx$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

$$ds = Rd\theta \quad Or \ \frac{d\theta}{ds} = \frac{1}{R}$$

$$\frac{d\theta}{dx} = \frac{1}{R} \dots\dots\dots\dots\dots\dots\dots (2)$$

$$\frac{d^2y}{dx^2} = \frac{1}{R}$$

But: $\frac{M_x}{I_{N.A}} = \frac{E}{R} = \frac{\sigma_{bending}}{y} \ Or \ \frac{1}{R} = \frac{M_x}{E.I_{N.A}}$

$$\therefore \frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M_x}{E.I_{N.A}} \ldots\ldots\ldots\ldots(3)$$

Equation (3) is called the differential equation of the elastic curve, and "$E.I_{N.A}$" is the flexural rigidity of the beam.

So: $\frac{d^2y}{dx^2} = \frac{M_x}{E.I_{N.A}}$

$$\rightarrow E.I_{N.A}\frac{d^2y}{dx^2} = M_X$$

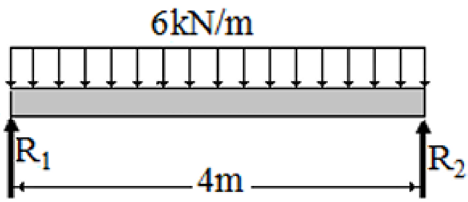
$$E.I_{N.A}\frac{dy}{dx} = \int M_X.dx + c_1$$

Here "M_X " is the moment equation in terms of "x", and "c_1 " is a constant evaluated from the given conditions of loading and/or the type of the beam.

$$E.I_{N.A}\frac{dy}{dx} = \int\int (M_X.dx)dx + c_1 x + c_2 \ldots\ldots\ldots\ldots(4)$$

Equation (4) is the equation of the elastic curve, and "c2" is another constant to be evaluated from the given conditions of the beam and its loading.

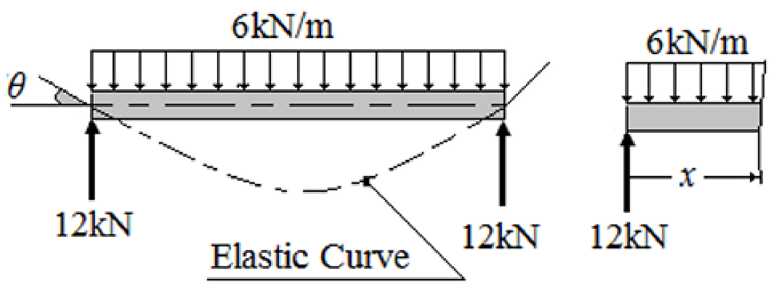
Example # 1 For the beam loaded as shown, determine the maximum deflection at the left support. Take EI = 86kN.m^2.



Solution: Firstly we must determine the values of both reactions. From symmetry, we can write:

$$R_1 = R_2 = \frac{6 * 4}{2} = 12\ kN$$

According to the loading of the beam, we need one section to write the equation of the moment. We can start from the left or from the right side of the beam, but in all our solved examples we will start from the left side.



$$E.I\frac{d^2y}{dx^2} = M_X$$

$$E.I\frac{d^2y}{dx^2} = 12X - 6X * \frac{X}{2} = 12X - 3X^2$$

$$E.I\frac{dy}{dx} = 6X^2 - X^3 + C_1 \dots\dots\dots\dots\dots\dots\dots..(1)$$

$$E.IY = 2X^3 - \frac{X^4}{4} + C_1X + C_2 \dots\dots\dots..(2)$$

Boundary conditions:X=0 →Y =0; Sub. in eq. (2) get that ($C_2 = 0$);

X=4 → Y=0; Sub. in eq. (2):

$$E.IY = 0$$

$$= 2 * 4^3 - \frac{4^4}{4} + C_1 4$$

$$\rightarrow C_2 = -16$$

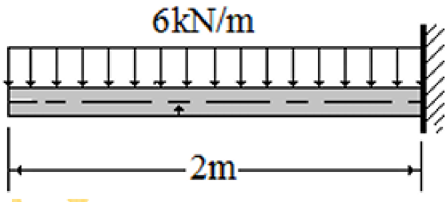
$$E.IY = 2X^3 - \frac{X^4}{4} - 16X \dots\dots\dots..(3)$$

Equation (3) is the elastic curve equation. To determine the maximum deflection (y_{max}): As shown, the maximum deflection will be at X =2 , so:

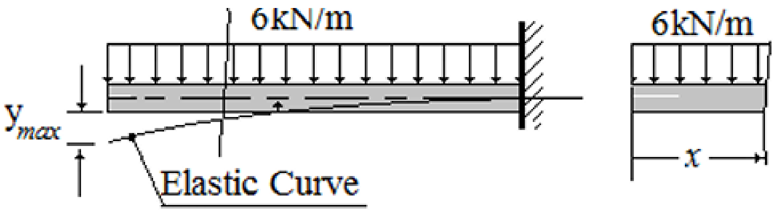
$$E.IY_{max} = 2 * 2^3 - \frac{2^4}{4} - 16 * 2 =$$

$$Y_{max} = -\frac{16 * 10^3}{86 * 10^3} = -0.186m = -186mm$$

Example # 2 A cantilever beam with a rectangular cross-sectional area of 100*200mm2 is loaded as shown. Determine the maximum deflection of the beam. Take E = 200GPa.



Solution: According to the loading of the beam, we need only one section to write the equation of the bending moment as shown in Fig. .



$$E.I\frac{d^2y}{dx^2} = M_X$$

$$E.I\frac{d^2y}{dx^2} = -6X * \frac{X}{2} = -3X^2$$

$$E.I\frac{dy}{dx} = -X^3 + C_1 \dots\dots\dots\dots\dots\dots\dots..(1)$$

$$E.IY = -\frac{X^4}{4} + C_1X + C_2 \dots\dots\dots..(2)$$

Boundary conditions:X=2 → $\frac{dy}{dx}$ =0; Sub. in eq. (1)

$$\rightarrow 0 = -\mathbf{2^3} + \mathrm{C1}$$

$$\mathrm{C1} = 8$$

X=2 → y=0; Sub. in eq. (2):

$$\boldsymbol{E.IY = 0}$$

$$= \mathbf{2 - \frac{2^4}{4} + 8 * 4 + C_2}$$

$$\rightarrow \boldsymbol{C_2 = -12}$$

$$\therefore \boldsymbol{E.IY = -\frac{X^4}{4} + 8X - 12 \ldots\ldots\ldots..(3)}$$

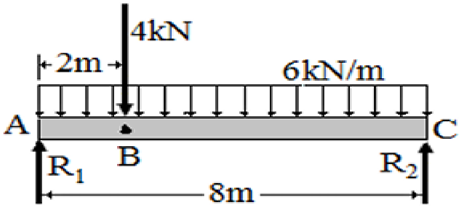
Equation (3) is the elastic curve equation. To determine the maximum deflection (y_{max}): As shown, the maximum deflection will be at X =0 , so:

$$\boldsymbol{E.IY_{max} = -0 + 0 - 12}$$

$$\boldsymbol{Y_{max} = -\frac{12 * 10^3}{(200 * 10^9)(\frac{0.1 * 0.2^3}{12})} = -0.9 * 10^{-3} m}$$

$$\boldsymbol{= -0.9mm}$$

Example # 3 For the beam loaded as shown, determine the deflection at point "B" under the concentrated load and the midspan deflection. Take EI = 86kN.m².



Solution:

Firstly, we must determine the values of both reactions.

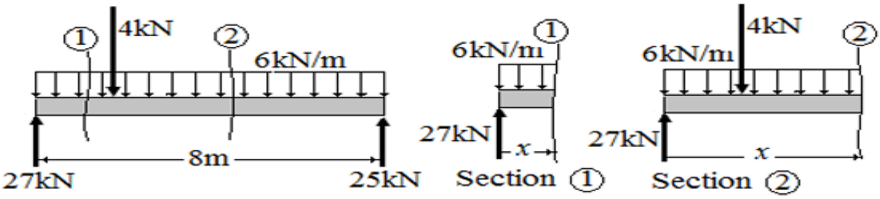
$$\sum M_{R_2} = 0; \qquad = 8R_1 - 4 * (8-2) - 6 * 8 * \frac{8}{2}$$

$$R_1 = 27kN$$

By using :$\sum F_y = 0; \qquad = 27 + R_2 - 4 - 6 * 8$

$$R_2 = 25kN$$

According to the change in loading condition along the beams entire length, there must be a corresponding change in the moment equation. We need two sections to write separate moment equations between each change of load points.



$$E.I\frac{d^2y}{dx^2} = M_X$$

$$M_{AB} = 27X - 6X * \frac{X}{2} = 27X - 3X^2(0 \le X \le 2)$$

$$M_{BC} = 27X - 6X * \frac{X}{2} - 4(X-2) = 27X - 3X^2 - 4(X-2)(0 \le X \le 8)$$

It is clear that $M_{BC}$is valid for M_{AB} if the term (X −2) is neglected for values of (X < 2) To remind these restrictions, we use pointed brackets (like < >)instead of the usual form of parentheses. Thus, we can write a single moment equation for the entire length of the beam. This equation is:

$$M_x = 27X - 3X^2 - 4(X-2)$$

$$E.I\frac{d^2y}{dx^2} = M_x \rightarrow E.I\frac{d^2y}{dx^2} = 27X - 3X^2 - 4(X-2)$$

The integration of all values in the pointed brackets must be carried out with respect to the whole bracket and will be neglected if its results is negative after substituting the value of "x".

$$E.I\frac{dy}{dx} = \frac{27x^2}{2} - X^3 - \frac{4(x-2)^2}{2} + C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$E.I\frac{dy}{dx} = \frac{27x^3}{6} - \frac{x^4}{4} - \frac{4(x-2)^3}{2*3} + C_1x + C_2 \ldots\ldots\ldots\ldots..(2)$$

Boundary conditions: X=0 →y =0; Sub. in eq. (2)

$$\rightarrow 0 = 0 - 0 - \frac{4(0-2)^3}{2*3} + C2$$

$$C2 = 0$$

X=8 → y=0; Sub. in eq. (1):

$$E.IY = 0$$

$$= \frac{278^3}{6} - \frac{8^4}{4} - \frac{4(8-2)^3}{2*3} + C_1 8 + C_2$$

$$\rightarrow C_1 = -142$$

$$\therefore E.IY = \frac{27x^3}{6} - \frac{x^4}{4} - \frac{4(x-2)^3}{2*3} - 142X \dots\dots\dots..(3)$$

To find the deflection at point "B", put (x=2) in Eq. 3, so:

$$E.IY_B = \frac{27*2^3}{6} - \frac{2^4}{4} - \frac{4(2-2)^3}{2*3} - 142*2$$

$$Y_B = \frac{252}{86*10^3} = -2.39*10^{-3}\, m$$

To find the midspan deflection, substitute (x=4) in Eq.3, so:

$$E.IY_{x=4} = \frac{27*4^3}{6} - \frac{2^4}{4} - \frac{4(4-2)^3}{2*3} - 142*4$$

$$Y_{x=4} = \frac{349.332}{86*10^3} = -4.062*10^{-3}\, m$$