Bisection method and false position method

Bisection method: It is one type of incremental search method which the interval is always divided in half. The location of the root is then determined as laying at the midpoint of the subinterval within which the sign change occurs. The following are the steps for actual computation:

Step #1: Choose the lower (x_l=a) and upper (x_u=b) guesses for the roots so that the function change sign.

Step #2: An estimate of the root (x_m=c) is determined by:

$$x_m = \frac{x_l + x_u}{2} \ldots\ldots\ldots\ldots(3.2)$$

Step #3: Make the following evaluation to determine which subinterval the root is:

1. $F(x_l)*f(x_m)<0$ the root lies in the lower subinterval
2. $F(x_l)*f(x_m)>0$ the root lies in the upper subinterval
3. $F(x_l)*f(x_m)=0$ the root equals (x_m)[terminate computation]

EXAMPLE:

Use the method of bisection to find the root of the equation, $f(x) = x^4 + 2x^3 - x - 1 = 0$ lying in the interval [0, 1] at the end of sixth iteration. How many iterations are required if the permissible error is Єs = 0.0005?

Sol. : Assume $x_l = a$, $x_u = b$ and $x_m = c$

The given interval is [a, b] = [0, 1]

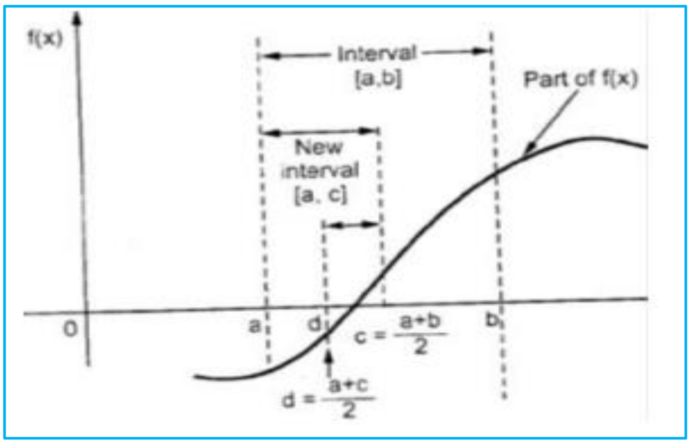
Iteration No. 1 a= 0, b= 1

$$\therefore \quad c = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

f(a) = f(0) = -1,

f(b) = f(1) = 1,

f(c) = f(0.5) = -1.1875



Since f(b) * f(c) < 0, root lies between 'b' and 'c'. Hence 'a' will be replaced by 'c'. Therefore new interval is [a, b] = [0.5, 1]

Iteration No. 2 a= 0.5, b =1 .'. c $=\frac{0.5+1}{2}=$ 0.75

f(a) = f(0.5)=-1.1875,
f(b) = f(l) = 1,
f(c) = f(0.75) = - 0.5898

Since f(b) * f(c)< 0, root lies between 'b' and 'c'. Hence 'a' will be replaced by 'c', and new interval will be [a, b] = [0.75, 1]

Iteration No. 3 a = 0.75, b = 1 .-. c $=\frac{0.75+1}{2}=$ 0.875

f(a) = f(0.75) = -0.5898,
f(b) = f(1) = 1,
f(c) = f(0.875) = 0.0510254
f(a) * f(c) < 0, root lies between 'a' and 'c'
'b' will be replaced by ' c' and New interval [a, b] = [0.75, 0.875]

Iteration No. 4 a= 0.75, b = 0.875 .-. c $=\frac{0.75+0.875}{2}=$ 0.8125

f(a) = f(0.75) = -0.5898
f(b) = f(0.875) = 0.0510254,
f(c) = f(0.8125) = - 0.3039398
f(b) * f(c) < 0, root lies between 'b' and 'c'
'a' will be replaced by 'c' and new interval [a, b] = [0.8125, 0.875]

Iteration No. 5 a = 0.8125, b = 0.875 , c $=\frac{0.8125+0.875}{2}=$ 0.84375

f(a) = f(0.8125) = - 0.3039398,
f(b) = f(0.875) = 0.0510254,
f(c) = f(0.84375) = - 0.1355733
f(b) * f(c) < 0, root lies between 'b' and 'c'
'a' will be replaced by 'c' and new interval [a, b] = [0.84375, 0.875]

Iteration No. 6 a = 0.84375, b = 0.875, c $=\frac{0.84375+0.873}{2}=$ 0.859375

f(a) = f(0.84375) = - 0.1355733,
f(b) = f(O.875) = 0.0510254,
f(c) = f(0.859375) = - 0.0446147
f(b) * f(c) < 0, root lies between 'b' and 'c'

'a' will be replaced by 'c' and new interval [a, b] = [0.859375, 0.875]

Hence root will be = $\frac{0.859375+0.875}{2}$ = 0.8671875

Thus root = 0.8671875 at the end of 6[th] iteration.

To determine number of iterations for permissible error Єs = 0.0005, for the given interval [a, b] = [0, 1]

Permissible error Єs = 0.0005

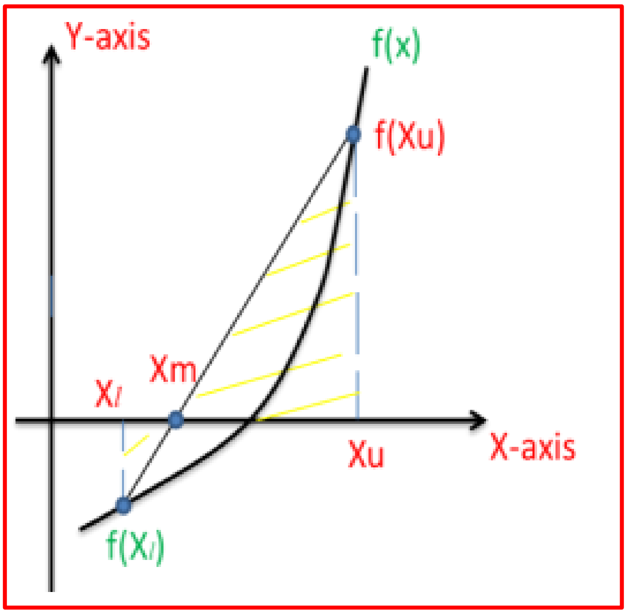
`Number of iterations are given from the equation:

$$n \geq \frac{\log(b-a)-\log(\epsilon_s)}{\log 2} \geq \frac{\log(1-0)-\log(0.0005)}{\log 2} \geq \frac{0-(-3.30103)}{0.30103} \geq 10.96$$

Hence (n = 11) iterations are required to get the error less than permissible error.

False position method or Regula Falsi method:

It is an improved version of the bisection method. An alternative way from halving the distance is to join the points by a straight line. The intersection of this line with the x-axis represents an improved estimate of the root. From the figure, the intersection of the straight line with the x-axis can be estimated as in the formula according to the two symmetrical triangles:



$$\frac{y}{x} = \frac{f(x_l)}{x_m - x_l} = \frac{f(x_u)}{x_m - x_u}$$

$$\therefore x_m = x_u - \frac{f(x_u)*(x_l - x_u)}{f(x_l) - f(x_u)}$$

EXAMPLE:

Find the root of $f(x) = e^x - 4x = 0$ using False position method, correct to three decimal places.

Solution: $f(x) = e^x - 4x = 0$, $f(0)=1$, $f(1)=-1.281718$

Hence, since $f(0) * f(1) < 0$, $\therefore$ *the root between* (0 *and* 1)

Let's take $x_l = x_0 = 0$, and $x_u = x_1 = 1$

Using the following relation we can find the next approximation of the root;

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Iteration No.1

$$x_2 = 1 - \frac{1-0}{(-1.281718) - 1} * (-1.281718) = 0.438266$$

$f(x_2) = e^{0.438266} - 4*(0.438266) = -0.203047$

Since $f(x_0) * f(x_2) < 0$, then the root lies between [0, 0.438266].

Hence we take initial approximation for second iteration as,

$x_1 = 0$, $x_2 = 0.438266$

Iteration No. 2:

With initial approximations of $x_1 = 0$ and $x_2 = 0.438266$ from the previous iteration, we find next approximation x_3 to the root as:

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$x_3 = 0.438266 - \frac{0.438266 - 0}{(-0.203047) - 1} * (-0.203047) = 0.364297$$

$f(x_3) = e^{0.364297} - 4*(0.364297) = -0.017686$

Since $f(x_1) * f(x_3) < 0$, root lies in the interval [0, 0.364297]

Iteration No.3:

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$x_4 = 0.364297 - \frac{0.364297 - 0}{(-0.017686) - 1}(-0.017686) = 0.357966$$

∴ $f(x_4) = -0.001447$

Since $f(x_2) * f(x_4) < 0$, root lies in the interval [0, 0.357966]

Iteration No. 4:

$x5$ = 0.357449

Since three decimal digits repeat in successive approximation, the approximation to the root is correct up to 3 decimal places.

∴ *Answer* = 0.357449

Home work:

Determine the real roots of the functions:

1. $F(x) = -0.9\,x^2 + 1.7x + 2.5$ [take x_l=2.8, x_u=3.0]
2. $F(x) = -2 + 6.2x - 4x^2 + 0.7x^3$ [take x_l=0.4, x_u=0.6]

Use bisection and false iteration method.