# Dimensional Consistency

Now that we have addressed units and dimensions and unit conversions, we can immediately make use of this information in a very practical and important application. A basic principle exists that equations must be dimensionally consistent. What the principle means is that each term in an equation must have the same net dimensions and units as every other term to which it is added or subtracted or equated. Consequently, dimensional considerations can be used to help identify the dimensions and units of terms or quantities in an equation. The concept of dimensional consistency can be illustrated by an equation that represents the pressure/volume/temperature behavior of a gas and is known as van der Waals' equation:

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

Inspection of the equation shows that the constant $a$ must have the units of $[(\text{pressure})(\text{volume})^2]$ because each term inside the first bracket must have units of pressure. If the units of pressure are atmospheres and those of volume are cubic centimeters, $a$ will have the units specifically of $[(\text{atm})(\text{cm})^6]$. Similarly, $b$ must have the same units as $V$, or in this particular case the units of cubic centimeters. If $T$ is in kelvin, what must be the units of R? Check your answer by looking up R inside the front cover of the book. All equations must exhibit dimensional consistency.

### Example 2.5 Dimensional Consistency

Your handbook shows that microchip etching roughly follows the relation

$$d = 16.2 - 16.2e^{-0.021t} \quad t < 200$$

where $d$ is the depth of the etch in microns [micrometers (μm)] and $t$ is the time of the etch in seconds. What are the units associated with the numbers 16.2 and 0.021? Convert this relation so that $d$ becomes expressed in inches and $t$ can be used in minutes.

**Solution**

After you inspect the equation that relates $d$ as a function of $t$, you should be able to reach a decision about the units associated with each term on the

(*Continues*)

### Example 2.5 Dimensional Consistency (*Continued*)

right-hand side of the equation. Based on the concept of dimensional consistency, both values of 16.2 must have the associated units of microns (μm). The exponential term must be dimensionless so that 0.021 must have units of $s^{-1}$. To carry out the specified unit conversion for this equation, look up suitable conversion factors inside the front cover of this book (i.e., convert 16.2 μm to inches and 0.021 $s^{-1}$ to $min^{-1}$).

$$d(\text{in.}) = \frac{16.2\ \mu\text{m}}{} \left| \frac{1\ \text{m}}{10^6\ \mu\text{m}} \right| \frac{39.37\ \text{in.}}{1\ \text{m}} \left[ 1 - \exp \frac{-0.021}{\text{s}} \left| \frac{60\ \text{s}}{1\ \text{min}} \right| \frac{t(\text{min})}{} \right]$$

$$d(\text{in.}) = 6.38 \times 10^{-4}(1 - e^{-1.26t(\text{min})})$$

As you proceed with the study of chemical engineering, you will find that groups of symbols may be put together, either by theory or based on experiment, that have no net units. Such collections of variables or parameters are called dimensionless or nondimensional groups. One example is the Reynolds number that arises in fluid mechanics:

$$\text{Reynolds number} = \frac{Dv\rho}{\mu} = N_{RE}$$

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where $D$ is the pipe diameter (e.g., cm), $\nu$ is the fluid velocity (e.g., cm/s), $\rho$ is the fluid density (e.g., g/cm$^3$), and $\mu$ is the viscosity [usually given in the units of centipoise, which itself has the units of g/(cm)(s)]. Introducing this consistent set of units for $D$, $\nu$, $\rho$, and $\mu$ into $D\nu\rho/\mu$, you will find that all the units cancel, resulting in a dimensionless number for the Reynolds number:

$$N_{RE} = \frac{D\nu\rho}{\mu} = \frac{\text{cm}}{}\left|\frac{\text{cm}}{\text{s}}\right|\frac{\text{g}}{\text{cm}^3}\left|\frac{(\text{cm})(\text{s})}{\text{g}}\right. = \text{a dimensionless quantity}$$

## Example:

***Dimensional Homogeneity and Dimensionless Groups***

A quantity $k$ depends on the temperature $T$ in the following manner:

$$k\left(\frac{\text{mol}}{\text{cm}^3\cdot\text{s}}\right) = 1.2\times10^5\exp\left(-\frac{20{,}000}{1.987T}\right)$$

The units of the quantity 20,000 are cal/mol, and $T$ is in K (kelvin). What are the units of $1.2\times10^5$ and 1.987?

Since the equation must be consistent in its units and exp is dimensionless, $1.2\times10^5$ should have the same units as $k$, mol/(cm$^3\cdot$s). Moreover, since the argument of exp must be dimensionless, we can write

$$\frac{20{,}000\ \text{cal}}{\text{mol}}\left|\frac{1}{T(\text{K})}\right|\frac{\text{mol}\cdot\text{K}}{1.987\ \text{cal}} \qquad \text{(All units cancel)}$$

The answers are thus

$$1.2\times10^5\ \text{mol/(cm}^3\cdot\text{s)} \quad \text{and} \quad 1.987\ \text{cal/(mol}\cdot\text{K)}$$

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## Example 2.6 Dimensional Consistency of an Equation

The following equation is proposed to calculate the pressure drop ($\Delta p$) across a length of pipe ($L$) due to flow through the pipe. Determine the dimensional consistency of this equation:

$$\Delta p = \frac{1}{2}v^2\left(\frac{L}{D}\right)f$$

where $v$ is the average velocity of the fluid flowing through the pipe, $D$ is the diameter of the pipe, and $f$ is a dimensionless coefficient called the friction factor, which is a function of the Reynolds number.

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### Solution

Let's substitute SI units appropriate for each term into the proposed equation, recognizing that pressure is force per unit area (see Table 2.1). What are the units of $\Delta p$? They are

$$\frac{\text{N}}{\text{m}^2} = \frac{(\text{kg})(\text{m})}{\text{s}^2}\bigg|\frac{}{\text{m}^2} \rightarrow \frac{\text{kg}}{(\text{s}^2)(\text{m})}$$

What are the net units of the right-hand side of the proposed equation?

$$\left(\frac{\text{m}}{\text{s}}\right)^2\bigg|\frac{\text{m}}{\text{m}} \rightarrow \frac{\text{m}^2}{\text{s}^2}$$

Therefore, because the units of the left-hand side of the equation do not match the units of the right-hand side, the proposed equation is not dimensionally consistent. By some research or inspection, it was determined that the proposed equation was missing a density term on the right-hand side of the equation; that is, the equation should be

$$\Delta p = \frac{1}{2}v^2\rho\left(\frac{L}{D}\right)f$$

With this modification, the units on the right-hand side of the equation become

$$\left(\frac{\text{m}}{\text{s}}\right)^2\bigg|\frac{\text{kg}}{\text{m}^3}\bigg|\frac{\text{m}}{\text{m}} \rightarrow \frac{\text{kg}}{(\text{s}^2)(\text{m})}$$

Therefore, if the density is included, this equation is shown to be dimensionally consistent.

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## Dimensional Analysis

Dimensional analysis is a method for helping determine how variables are related and for simplifying a mathematical model. Dimensional analysis alone does not give the exact form of an equation, but it can lead to a significant reduction of the number of variables. It is based on two assumptions:

**1. Physical quantities have dimensions (fundamental are mass M, length L, and time T). Any physical quantity has a dimension which is a product of powers of the basic dimensions M, L and T.**

**2. Physical laws are unaltered when changing the units measuring the dimensions.**

Units must be taken into consideration when collecting the data as well as when making the list of factors impacting the model and when testing the model. You must check that all the equations in a model are dimensionally consistent.

The concept of dimensional consistency (the second assumption above) is related to **dimensional homogenicity.** For example, the equation t $=\sqrt{2s/g}$ that describes the time a bodyfalls from a distance s under gravity is true in all systems, whereas the equation t $=\sqrt{\frac{s}{16.1}}$ is not dimensionally homogeneous because it depends on a particular systems (units of g are neglected so the units of the left and the right side of the equation do not match).

The following table of dimensions of physical entities in the $MLT$ system can be useful.

| | | | |
|---|---|---|---|
| Mass | $M$ | Angular acceleration | $T^{-2}$ |
| Length | $L$ | Momentum | $MLT^{-1}$ |
| Time | $T$ | Angular momentum | $ML^2T^{-1}$ |
| Velocity | $LT^{-1}$ | Density | $ML^{-3}$ |
| Acceleration | $LT^{-2}$ | Viscosity | $ML^{-1}T^{-1}$ |
| Force | $MLT^{-2}$ | Pressure | $ML^{-1}T^{-2}$ |
| Energy, work, heat, torque, entropy | $ML^2T^{-2}$ | Surface tension | $MT^{-2}$ |
| Frequency, angular velocity | $T^{-1}$ | Power | $ML^2T^{-3}$ |

**Example 1.** Find a model describing the terminal velocity of a particle that falls under gravity through a viscous fluid.

When a particle falls through a viscous fluid, the drag force counteracts with the acceleration caused by the gravity and after some time, the velocity stops increasing. This is called the terminal velocity. We can assume that the velocity $v$ depends on the particle's diameter $D$, the viscosity $\mu$ and the acceleration $g$. Also, we will assume that the velocity is directly proportional to the difference between the density $\rho_1$ of the particle and the density $\rho_2$ of the fluid. Thus,

$$v = kD^a\mu^b g^c(\rho_1 - \rho_2)$$

where $k$ is the proportionality (dimensionless) constant and $a, b$ and $c$ are unknown numbers that we will determine using the dimensional analysis.

Note that the density have unit $ML^{-3}$, the gravity has the unit of acceleration $LT^{-2}$ and the viscosity has the unit $ML^{-1}T^{-1}$.

Consider the dimensions of both sides:

$$\begin{aligned} LT^{-1} &= L^a(ML^{-1}T^{-1})^b(LT^{-2})^c(ML^{-3}) \\ &= M^{b+1}L^{a-b+c-3}T^{-b-2c} \end{aligned}$$

Equating the exponents of $M$, $L$ and $T$ on both sides we obtain a system of three equations in three unknowns.

$$\begin{aligned} 0 &= b+1 \\ 1 &= a-b+c-3 \\ -1 &= -b-2c \end{aligned}$$

The solution of the system is $a = 2$, $b = -1$, and $c = 1$. Thus the equation describing the velocity is

$$v = \frac{kD^2g(\rho_1 - \rho_2)}{\mu}.$$

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Note that if an expression involving functions as $e^{at}$ or $\sin(at)$ where $t$ is time appears in a model, then $a$ must have dimensions $T^{-1}$.

Also, recall that the dimensions of the derivative are the ration of the dimensions. For example, the derivative of pressure with respect to time has the units

$$\left[\frac{dp}{dt}\right] = \frac{[p]}{[t]} = \frac{ML^{-1}T^{-2}}{T} = ML^{-1}T^{-3}$$

Analogous rule applies to the partial derivatives as well.

Example :

**a)** Determine the dimensions of the following quantities in M-L-T system 1- force 2-pressure 3- work 4- power 5- surface tension 6- discharge 7- torque 8- momentum.

**b)**Check the dimensional homogeneity of the following equations

$$u = \sqrt{\frac{2g(\rho_m - \rho)}{\rho}} \qquad Q = \frac{8}{15} Cd \tan\frac{\theta}{2}\sqrt{2g}\, z^{5/2}$$

***Solution:a)***

**1- F = m.g (kg.m/s$^2$)** $\equiv [MLT^{-2}]$

**2- P=F/A ≡ [(MLT$^{-2}$) (L$^{-2}$)] (Pa)** $\equiv [ML^{-1}T^{-2}]$

**3- Work = F.L ≡ [(MLT$^{-2}$) (L)] (N.m)** $\equiv [ML^{2}T^{-2}]$

**4- Power = Work/time ≡ [(ML$^2$T$^{-2}$) (T$^{-1}$)] (W)** $\equiv [ML^{-1}T^{-2}]$

**5- Surface tension = F/L ≡ [(MLT$^{-2}$) (L$^{-1}$)] (N/m)** $\equiv [ML^{-1}T^{-2}]$

**6- Discharge (Q) m$^3$/s** $\equiv [L^{3}T^{-1}]$

**7- Torque (Γ) = F.L ≡ [(MLT$^{-2}$) (L)] N.m** $\equiv [ML^{2}T^{-2}]$

**8- Moment = m.u L)] N.m** $\equiv [ML^{2}T^{-2}]$

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**b)**

**L.H.S. u ≡ $[LT^{-1}]$** R.H.S. u $\equiv \left[\frac{LT^{-2}(ML^{3})}{ML^{-3}}\right]^{1/2} \equiv [LT^{-1}]$

Since the dimensions on both sides of the equation are same, therefore the equation is dimensionally homogenous.

**L.H.S.u ≡ $[L^{3}T^{-1}]$ R.H.S. $(LT^{-2})\,(L)^{5/2} \equiv [L^{3}T^{-1}]$**

This equation is dimensionally homogenous.

Home Work:

Explain what dimensional consistency means in an equation?