Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

Subject Name: CAD 2

4th Class, Second Semester

Subject Code: [MU0114205]

Academic Year: 2024-2025

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Lecture No.4

Lecture Title: [Beam Analysis]

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One Dimensional Polynomial Shape Function

A general one dimensional polynomial shape function of nth Order is given by,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \alpha_{n+1} x^n$$

In matrix form $u = [G]\{\alpha\}$

where $$[G] = [1, x, x^2 \dots x^n]$$

and $$\{\alpha\}^T = [\alpha_1 \; \alpha_2 \; \alpha_3 \dots \alpha_{n+1}]$$

Thus in one dimensional n^{th} order complete polynomial there are $m = n + 1$ terms.

Two Dimensional Polynomial Shape Function

A general form of two dimensional polynomial model is

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 \dots + \alpha_m y^n$$

$$v(x, y) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \dots + \alpha_{2m} y^n$$

or $$\{\delta\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}\{\alpha\}$$

where $$G_1 = [1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2 \;\; x^3 \dots y^n]$$

$$\{\alpha\}^T = [\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4 \dots \alpha_{2m}]$$

It may be observed that in two dimensional problem, total number of terms m in a complete nth degree polynomial is

$$m = \frac{(n+1)(n+2)}{2}$$

For first order complete polynomial $n = 1$,

$$\therefore m = \frac{(1+1)(1+2)}{2} = 3$$

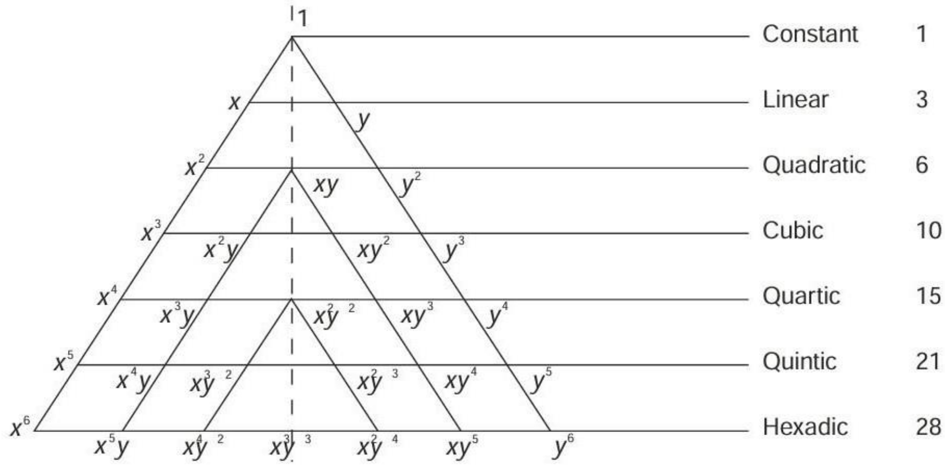
The first three terms are $\alpha_1 + \alpha_2 x + \alpha_3 y$

Similarly for $n = 2$, $m = \frac{(2+1)(2+2)}{2} = 6$

and we know the first six terms are,

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

Another convenient way to remember complete two dimensional polynomial is in the form of Pascal Triangle shown in Fig. 5.2



Pascal triangle

Three Dimensional Polynomial Shape Function

A general three dimensional shape function of *n*th order complete polynomial is given by

$$u(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \ldots + \alpha_m x^{n-1} z$$
$$v(x, y, z) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} z + \alpha_{m+5} x^2 + \ldots + \alpha_{2m} x^{n-1} z$$
$$w(x, y, z) = \alpha_{2m+1} + \alpha_{2m+2} x + \alpha_{2m+3} y + \alpha_{2m+4} z + \ldots + \alpha_{3m} x^{n-1} z$$

or

$$\delta(x, y, z) = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_1 \end{bmatrix} \{\alpha\} \quad ..$$

Where $G_1 = [1\ x\ y\ z\ x^2\ xy\ y^2\ yz\ z^2\ zx \ldots z^n\ z^{n-1}x \ldots\ zx^{n-1}]$

and
$$\{\alpha\}^T = [\alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_{3m}]$$

It may be observed that a complete *n*th order polynomial in three dimensional case is having number of terms *m* given by the expression

$$m = \frac{(n+1)(n+2)(n+3)}{6}$$

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Thus when $n = 1$, $m = \dfrac{(1+1)(1+2)(1+3)}{6} = 4$

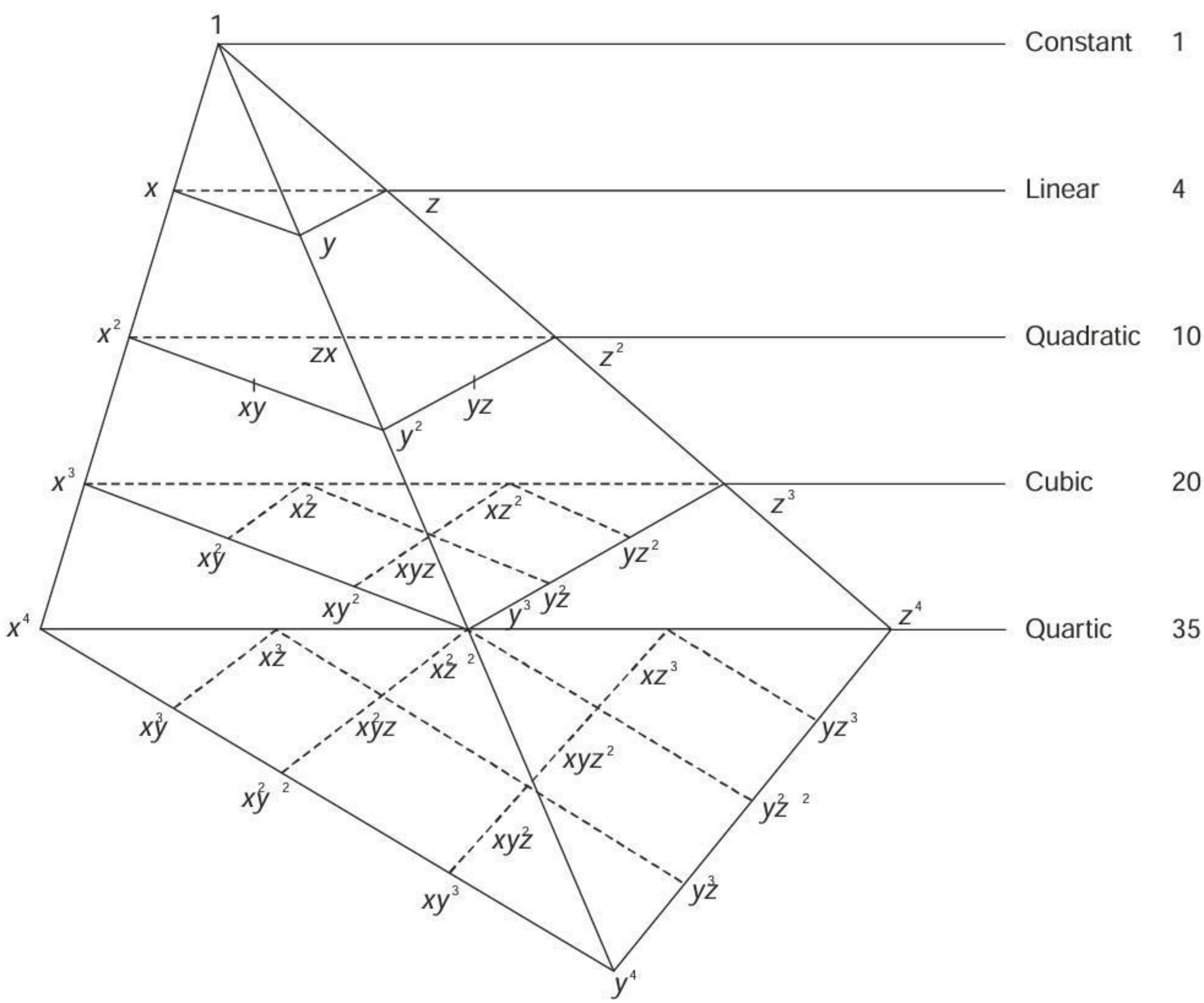
i.e. $$\alpha_1 + \alpha_2\, x + \alpha_3\, y + \alpha_4\, z$$

For $n = 2$, $m = \dfrac{(2+1)(2+2)(2+3)}{6} = 10$

Thus second degree complete polynomial is

$$\alpha_1 + \alpha_2\, x + \alpha_3\, y + \alpha_4\, z + \alpha_5\, x^2 + \alpha_6\, xy + \alpha_7\, y^2 + \alpha_8\, yz + \alpha_9\, z^2 + \alpha_{10}\, zx$$

Complete polynomial in three dimensions may be expressed conveniently by a tetrahedron as shown in



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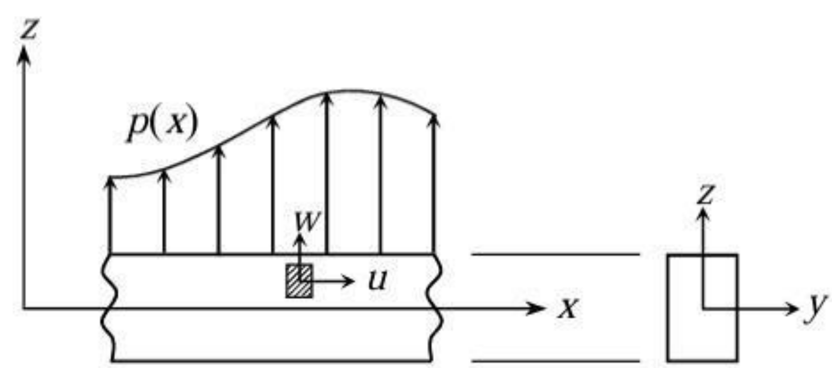
Beam Analysis

Basic Equations

Differential Equation

A beam that lies in the *x*-direction with its cross section in the *y*-*z* plane is shown in the figure. The beam is subjected to a distributed load $p(x)$ causing the deflection of w in the *z*-direction and the displacement of u in the *x*-direction.



If beam deflection is small, the small deformation theory stating that the plane sections before and after deflection remain plane is applied. This lead to the relation such that the displacement u can be written in form of the deflection w as $u = -z\,\partial w/\partial x$. In addition, if the beam is long and slender, the deflection w may be assumed to vary with x only, i.e., $w = w(x)$. These two assumptions yield to the equilibrium equation of the beam deflection as,

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = p$$

Related Equations

The stress σ_x along the axial *x*-coordinate of the beam varies with the strain ε_x according to the Hook's law as,

$$\sigma_x = E\,\varepsilon_x$$

Since the strain is related to the displacement and deflection as,

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

Then, the stress can be determined from the deflection as,

$$\sigma_x = -E\,z\frac{\partial^2 w}{\partial x^2}$$

For a typical beam in a three-dimensional frame structure, its deflection may be in a direction other than the *z*-coordinate. In addition, the beam may be twisted by torsion caused by the applied loads or affected by other members. These influences must be considered and included for the analysis of three dimensional beam structures.

Finite Element Method

Finite Element Equations

Finite element equations can be derived directly from the beam governing differential equation by using the method of weighted residuals. Detailed derivation can be found in many finite element textbooks including the one written by the same author. The derived finite element equations are in the form,

$$[K]\{\delta\} = \{F\}$$

where $[K]$ is the element stiffness matrix; $\{\delta\}$ is the element vector containing nodal unknowns of deflections and slopes; and $\{F\}$ is the element vector containing nodal forces and moments. These element matrices depend on the selected beam element types as explained in the following section.

Element Types

The basic beam bending element with two nodes is shown in the figure. Each node has two unknowns of the deflection w and slope θ.

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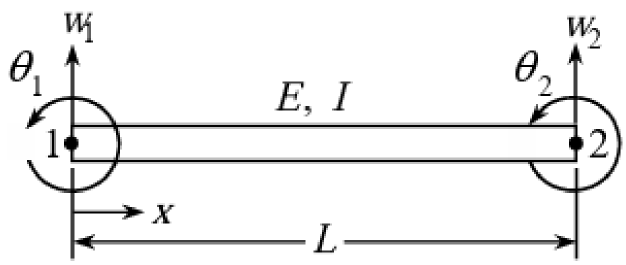

Distribution of the deflection w is assumed in the form,

$$w(x) = \lfloor N_1 \;\; N_2 \;\; N_3 \;\; N_4 \rfloor \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \underset{(1\times 4)}{\lfloor N(x) \rfloor} \underset{(4\times 1)}{\{\delta\}}$$

where the element interpolation functions are,

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \quad ; \quad N_2 = x\left(\frac{x}{L} - 1\right)^2$$

$$N_3 = \left(\frac{x}{L}\right)^2\left(3 - 2\frac{x}{L}\right) \quad ; \quad N_4 = \frac{x^2}{L}\left(\frac{x}{L} - 1\right)$$

These interpolation functions lead to the finite element equations as,

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} + \frac{p_0 L}{2}\begin{Bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{Bmatrix}$$

where F_1 and F_2 are the forces, while M_1 and M_2 are the moments, at node 1 and 2, respectively. The last vector contains the nodal forces and moments from the distributed load p_0 which is uniform along the element length.

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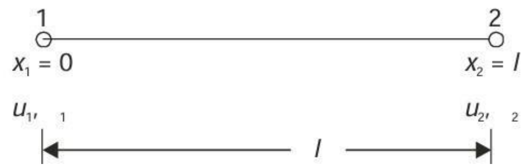
Example Using polynomial functions (generalized coordinates) determine shape functions for a two noded beam element.



$$\{\delta\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \Rightarrow \quad \text{where} \quad \theta_1 = \frac{\partial w_1}{\partial x} \quad \text{and} \quad \theta_2 = \frac{\partial w_2}{\partial x}$$

Since there are four nodal values, we select polynomial with four constants. Thus

$$w = \alpha_1 + \alpha_2\, x + \alpha_3\, x^2 + \alpha_4\, x^3$$

Equation 5.13 satisfies compatibility and completeness requirement. Now,

$$\theta = \frac{\partial w}{dx} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

For convenience we select local coordinate system.

i.e.,

$$x_1 = 0$$

$$x_2 = l$$

$$\therefore\ w_1 = \alpha_1$$

$$\theta_1 = \alpha_2$$

$$w_2 = \alpha_1 + \alpha_2\, l + \alpha_3\, l^2 + \alpha_4\, l^3$$

$$\theta_2 = \alpha_2 + 2\alpha_3 l + 3\alpha_4 l^2$$

i.e.,

$$\{\delta\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{3l^4 - 2l^4} \begin{bmatrix} l^4 & 0 & -3l^2 & 2l \\ 0 & l^4 & -2l^3 & l^2 \\ 0 & 0 & 3l^2 & -2l \\ 0 & 0 & -l^3 & l^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore \; w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} & x - \frac{2x^2}{l} + \frac{x^3}{l^2} & \frac{3x^2}{l^2} - \frac{2x^3}{l^3} & \frac{-x^2}{l} + \frac{x^3}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \{\delta\}_e = [N] \{\delta\}_e$$

where $[N] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$

and $N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \quad N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

Variation of these function is shown in Fig. 5.6 (b) (Note that at node 1, N_1=1,

$N_2 = N_3 = N_4 = 0$, and $\frac{\partial N_2}{\partial x} = 1$, $\frac{\partial N_1}{\partial x} = \frac{\partial N_3}{\partial x} = \frac{\partial N_4}{\partial x} = 0$ similarly at node 2,

$N_1 = N_2 = N_4 = 0$, $N_3 = 1$ and $\frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial x} = \frac{\partial N_3}{\partial x} = 0$ and $\frac{\partial N_4}{\partial x} = 1$,