Application of Derivatives : تطبيقات المشتقات

L-Hôpital's Rule : Indeterminate Forms :

قاعدة لوبيتال ، الأشكال / الكميات الغير محددة (غير معروفة)

Before starting up with this lecture, let us address the spelling of "L'Hôpital". It is the name of French mathematician "Guillaume de l'Hôpital". In the 17th & 18th centuries, the name was commonly spelled as "l'Hospital", & the mathematician himself spelled his name that way. However, French spellings have been altered, the silent "s" have been removed & replaced with the circumflex over the preceding vowel "o".

Indeterminate Forms : الكميات / الأشكال الغير محددة

In mathmatics, there are several types of quantities that we don't Know their exact value! Such as : $\frac{0}{0}$, $\frac{\mp\infty}{\mp\infty}$, $0*(\mp\infty)$, 1^{∞} , 0^{0} , ∞^{0} , $\infty-\infty$

Back to the lecture of limits we studied last course, we learn how to deal with the following limits :

① $\lim_{x\to 4} \frac{x^2-16}{x-4} \xrightarrow[\text{sub}]{\text{direct}} \frac{4^2-16}{4-4} = \frac{0}{0}$ (indeterminate quantity)

② Factorize the numerator $\rightarrow \lim_{x\to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}}$

$$= \lim_{x\to 4} (x+4) = 4+4 = \boxed{8}$$

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② $\lim\limits_{x\to\infty} \dfrac{4x^2-5x}{1-3x^2} \xrightarrow[\text{sub}]{\text{① direct}} \dfrac{\infty}{-\infty}$ (indeterminate quantity)

② Factor out x^2 from numerator & denominator $\longrightarrow \lim\limits_{x\to\infty} \dfrac{x^2(4-\frac{5}{x})}{x^2(\frac{1}{x^2}-3)}$

$$= \lim_{x\to\infty} \frac{4-\frac{5}{x}}{\frac{1}{x^2}-3} = \frac{4-\cancel{\frac{5}{\infty}}^{\,0}}{\cancel{\frac{1}{(\infty)^2}}^{\,0}-3} = \frac{4}{-3} = \boxed{-\frac{4}{3}}$$

But, what about the following two limits:

③ $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$ (can't be factored)

④ $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \dfrac{\infty}{\infty}$ (can't factor x^2 out of numerator)

In last two limits no tricks will work with them !!

So, to deal with such problems of limits L'Hôpital Rules comes into play.

L'Hôpital Rule : قاعدة لوبيتال

If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\mp\infty}{\mp\infty}$

where a = real number, $\mp\infty$

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \longleftarrow \text{L'Hôpital's Rule}$$

اذن تستخدم قاعدة لوبيتال لكل مشكلة وجود $\frac{0}{0}$ أو $\frac{\mp\infty}{\mp\infty}$ عند ايجاد الغاية لدالة كسرية

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Examples :-

① $\lim_{x \to 4} \frac{x^2-16}{x-4} \longrightarrow$ direct substitution $\longrightarrow \frac{4^2-16}{4-4} = \boxed{\frac{0}{0}}$

(USD) "un-defined sol"

As we got $\frac{0}{0}$, then we can use L'Hôpital's Rule

$\lim_{x \to 4} \frac{2x}{1} \longrightarrow \lim_{x \to 4} 2x = 2 \times 4 = \boxed{8}$ Ans

② $\lim_{x \to \infty} \frac{4x^2-5x}{1-3x^2} \longrightarrow$ direct sub $\longrightarrow \frac{4(\infty)^2-5(\infty)}{1-3(\infty)^2} = \boxed{\frac{\infty}{-\infty}}$ (UDS)

As we got $\frac{\infty}{-\infty}$, we can use L'Hôpital's Rule.

$\lim_{x \to \infty} \frac{4 \times 2x - 5}{1 - 3 \times 2x} = \lim_{x \to \infty} \frac{8x-5}{1-6x} \longrightarrow$ direct sub $\longrightarrow \frac{8(\infty)-5}{1-6(\infty)}$

$= \boxed{\frac{\infty}{-\infty}}$ (USD)

As we got $\frac{\infty}{-\infty}$, again we can use L'Hôpital's Rule

$\lim_{x \to \infty} \frac{\cancel{8}^{4}}{\cancel{-6}_{-3}} = \boxed{\frac{4}{-3}}$ Ans

③ $\lim_{x \to 0} \frac{\sin x}{x} \longrightarrow$ direct sub $\longrightarrow \frac{\sin 0}{0} = \boxed{\frac{0}{0}}$ (USD)

By using L'Hôpital's Rule

$\lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = \boxed{1}$ Ans

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④ $\lim_{t \to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3} \xrightarrow[\text{sub.}]{\text{direct}} \frac{5(1)^4 - 4(1)^2 - 1}{10 - 1 - 9(1)^3} = \frac{5-5}{10-10} = \boxed{\frac{0}{0}}$ (UDS)

If we were really good at Factoring we could Factor the numerator & denominator, simplify & take the limit. However, that's going to be more work than just using "L'Hôpital Rule"

$$\lim_{t \to 1} \frac{5 \times 4t^3 - 4 \times 2t}{-1 - 9 \times 3t^2} = \lim_{t \to 1} \frac{20t^3 - 8t}{-1 - 27t^2} = \frac{20 - 8}{-1 - 27}$$

$$= \frac{\cancel{12}^{\,3}}{\cancel{-28}_{\,-7}} = \boxed{-\frac{3}{7}} \quad \underline{\text{Ans}}$$

⑤ $\lim_{x \to \infty} \frac{e^x}{x^2} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^\infty}{(\infty)^2} = \boxed{\frac{\infty}{\infty}}$ (USD)

By applying "L'Hôpital Rule"

$$\lim_{x \to \infty} \frac{e^x}{2x} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^\infty}{2 \times \infty} = \boxed{\frac{\infty}{\infty}} \quad \text{(USD)}$$

By applying "L'Hôpital Rule" one more time,

$$\lim_{x \to \infty} \frac{e^x}{2} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^\infty}{2} = \frac{\infty}{2} = \boxed{\infty} \quad \underline{\text{Ans}}$$

Note① Sometimes we need to apply L'Hôpital's Rule more than once as we have seen in examples ② & ⑤

Ex: بعض الأحيان تحتاج لاستخدام قاعدة لوبيتال اكثر من مرة لحل السؤال كما تلاحظ المثال ② و ⑤

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(6) $\lim_{x \to 0^+} x \ln x \xrightarrow[\text{sub}]{\text{direct}} 0 \ln 0 = \boxed{0 \times -\infty}$ (UDS)

In this example, by direct substitution, we got $(0 \times -\infty)$ which is one of indeterminate forms
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we stated before, It is not quotient, So, we can't use L'Hôpital Rule directly unless we turn it into a fraction as follows,

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} \xrightarrow[\text{sub}]{\text{direct}} \frac{\ln 0}{1/0} = \boxed{\frac{-\infty}{\infty}}$$

Now, we can use "L'Hopital Rule",

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/\cancel{x}}{-1/x^{\cancel{2}\,1}} = \lim_{x \to 0^+} \frac{1}{-1/x}$$

$$= \lim_{x \to 0^+} (-x) = \boxed{0} \quad \text{Ans}$$

Note(2)

From example (6), we use the fact that we can always write a product of functions as a quotient by doing one of the following

$$f(x)\, g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)\, g(x) = \frac{g(x)}{1/f(x)}.$$

we can use "Note (2)" to turn any limit in the form "$0 \times \mp\infty$" into limit in the form $\frac{0}{0}$ or $\frac{\mp\infty}{\mp\infty}$

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(7) Evaluate the Following limit

$$\lim_{x\to-\infty} x\,e^{x}$$

Sol.

By direct substitution ; بالتعويض المباشر

$$\lim_{x\to-\infty} x\,e^{x} = (-\infty)\cdot \cancel{e^{-\infty}}^{\,0} = (-\infty)(0)$$

∴ حسب الملاحظة ② يمكننا تحويل هذه الحالة الى حالة كسرية (بسط ومقام) لكن لدينا خيارين هما اما ان نضع e^{x} في البسط و x بالمقام أو العكس، لنبدأ مع الخيار الاول

(1) $$\lim_{x\to-\infty} \frac{e^{x}}{1/x} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^{-\infty}}{1/-\infty} = \frac{0}{0} \quad (\text{L'Hôpital Rule})$$

$$\therefore \lim_{x\to-\infty} \frac{e^{x}}{-1/x^{2}} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^{-\infty}}{-1/(-\infty)^{2}} = \frac{0}{0} \quad (\text{L'Hôpital Rule again})$$

$$\lim_{x\to-\infty} \frac{e^{x}}{2/x^{3}} \xrightarrow[\text{sub.}]{\text{direct}} \frac{e^{-\infty}}{2/(-\infty)^{3}} = \frac{0}{0} \quad (\text{L'Hôpital Rule again})$$

$$\lim_{x\to-\infty} \frac{e^{x}}{-6/x^{4}} \xrightarrow[\text{sub}]{\text{direct}} \frac{e^{-\infty}}{-6/(-\infty)^{4}} = \frac{0}{0}$$

∴ This option dosen't seem to be getting us anywhere

∴ هذا الخيار يبدو لا يؤدي الى نتيجة، فمتى كل مرة يؤدي استخدام قاعدة لوبيتال الى $\boxed{\frac{0}{0}}$ وفي الحقيقة هنا استخدام قاعدة لوبيتال تعطي نتيجة اسوء في كل مرة نستخدمها. هنا علينا أن نعمل العكس بوضع دالة e^{x} في المقام و x في البسط، كما في التالي:

(2) $$\lim_{x\to-\infty} \frac{x}{1/e^{x}} = \lim_{x\to-\infty} \frac{x}{e^{-x}} \xrightarrow[\text{sub.}]{\text{direct}} \frac{-\infty}{e^{\infty}} = \boxed{\frac{-\infty}{\infty}}$$

هنا تحقق شرط استخدام قاعدة لوبيتال :

$$\therefore \lim_{x\to-\infty} \frac{1}{-e^{-x}} \xrightarrow[\text{sub}]{\text{direct}} \frac{1}{-e^{\infty}} = \frac{1}{-\infty} = \boxed{0} \quad \underline{\text{Ans}}$$

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(8) Evaluate the Following limit

$$\lim_{x\to\infty} x^{1/x}$$

Sol:

The direct substitution will give the indeterminate Form (∞^0)

we do the Follows to get indeterminate Form $\frac{0}{0}$ or $\frac{\mp\infty}{\mp\infty}$

let $\left[y = \lim_{x\to\infty} x^{1/x} \right] \times \ln$

$$\ln y = \lim_{x\to\infty} \ln x^{1/x}$$

$$\ln y = \lim_{x\to\infty} \frac{1}{x} \ln x = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\ln\infty}{\infty}$$

$$= \boxed{\frac{\infty}{\infty}} \quad (UDS)$$

Apply L'Hôpital Rule, yields;

$$\ln y = \lim_{x\to\infty} \frac{1/x}{1} \xrightarrow[x=\infty]{PV} \frac{1/\infty}{1} = \frac{0}{1} = 0$$

$$\therefore [\ln y = 0] \times e \Longrightarrow e^{\ln y} = e^{0}$$

$$\therefore y = e^{0} = 1$$

$$\therefore \lim_{x\to\infty} x^{1/x} = \boxed{1} \quad \text{Ans}$$

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(9) Evaluate the Following limit:

$$\lim_{x \to -\infty} x^2 e^{x}$$

Sol

By direct substitution, yields, بواسطة التعويض المباشر

$$(-\infty)^2 e^{-\infty} = \frac{\infty}{e^{\infty}} = \boxed{\frac{\infty}{\infty}} \quad \text{"UDS"}$$

As the answer is indeterminate $\frac{\infty}{\infty}$, then we can use L'Hôpital's Rule, but we need to rewrite the function in the form of quotient (كسر)

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} \xrightarrow[\text{Rule}]{\text{L'Hôpital}} \lim_{x \to -\infty} \frac{2x}{-e^{-x}} \xrightarrow[\text{Rule}]{\text{L'Hôpital}} \lim_{x \to -\infty} \frac{2}{e^{-x}}$$

$$\xrightarrow[\text{sub}]{\text{dir}} \frac{2}{e^{\infty}} = \frac{2}{\infty} = \boxed{0} \quad \text{Ans}$$

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(10) Evaluate $\lim_{x \to 0^+} \sin x \ln x$

Sol

By direct substitution, yields,

$$\sin(0) \ln 0 = 0 * (-\infty) \quad \text{UDS}$$

As the answer is indeterminate $0(-\infty)$, we can't use L'Hôpital's Rule unless we able to re-write the eq. to give either $\frac{0}{0}$ or $\frac{\mp\infty}{\mp\infty}$, the suitable way is;

$$\lim_{x \to 0^+} \frac{\ln x}{1/\sin x} \xrightarrow[\text{sub}]{\text{dir}} \frac{\ln 0}{\frac{1}{\sin 0}} = \frac{\infty}{\frac{1}{0}} = \boxed{\frac{\infty}{\infty}} \quad \text{(now can use) L' Hôpital's Rule}$$

$$\xrightarrow[\text{Rule}]{\text{L'H}} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x \to 0^+} \left[\frac{1}{x} * \frac{-\sin^2 x}{\cos x}\right]$$

$$= \lim_{x \to 0^+} \left[-\frac{\sin x}{x} * \frac{\sin x}{\cos x}\right] \xrightarrow[\text{sub}]{\text{dir}} -1 * \frac{\sin 0}{\cos 0}$$

$$= -1 * 0 = \boxed{0} \quad \text{Ans}$$

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(11) Evaluate $\lim_{x\to\infty} [x - \ln x]$

Sol/

Direct substitution yields,

$$\infty - \ln\infty = \infty - \infty \quad \text{"UDS"}$$

هذا السؤال مشابه للسؤال السابق، نحتاج اعادة كتابة الدالة لتكون دالة كسرية لكي تستطيع استخدام قاعدة لوبيتال معها، والطريقة المناسبة هو بواسطة استخدام طريقة المرافق "Conjugate"

$$\lim_{x\to\infty} [x - \ln x] \times \frac{(x+\ln x)}{(x+\ln x)} = \lim_{x\to\infty} \frac{x^2 - (\ln x)^2}{x + \ln x}$$

Direct substitution yields

$$\lim_{x\to\infty} \frac{x^2 - (\ln x)^2}{x + \ln x} \xrightarrow[\text{sub}]{\text{direct}} \frac{(\infty)^2 - (\ln\infty)^2}{\infty + \ln\infty} = \boxed{\frac{\infty}{\infty}} \quad \text{(Can use L'H. Rule)}$$

$$\lim_{x\to\infty} \frac{[2x - 2\ln x \times \frac{1}{x}]}{[1 + \frac{1}{x}]} = \lim_{x\to\infty} \frac{2x - \frac{2\ln x}{x}}{1 + \frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{2x^2 - 2\ln x}{\cancel{x}}}{\frac{x+1}{\cancel{x}}} = \lim_{x\to\infty} \frac{2x^2 - 2\ln x}{x+1} \xrightarrow[\text{sub}]{\text{dir}} \frac{\infty}{\infty} \quad \text{(L-H Rule)}$$

$$= \lim_{x\to\infty} \frac{4x - \frac{2}{x}}{1} = \lim_{x\to\infty} [4x - \frac{2}{x}] \xrightarrow[\text{sub}]{\text{dir}} [4(\infty) - \frac{2}{\infty}]$$

$$= \infty - 0 = \boxed{\infty} \quad \text{Ans}$$

Questions For Discussions — أسئلة للمناقشة

(1) $\lim_{x\to 0} [\csc x - \cot x]$

(2) Proof that $\lim_{x\to\infty} \sqrt{x^2+4x} - x = 2$

(3) $\lim_{x\to\infty} (1 + \frac{2}{x})^x$

(4) $\lim_{x\to 0} (1-3x)^{1/x}$

(5) $\lim_{x\to\infty} \frac{\ln x}{x}$

(6) $\lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$