Chapter 4: Vectors, Matrices, and Linear Algebra

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Linear Algebra is strikingly similar to the algebra you learned in high school, except that in the place of ordinary single numbers, it deals with vectors. Many of the same algebraic operations you're used to performing on ordinary numbers (a.k.a. scalars), such as addition, subtraction and multiplication, can be generalized to be performed on vectors. We'll better start by defining what we mean by *scalars* and *vectors*.

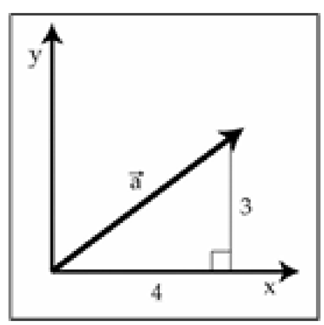
Definition: A scalar is a number. Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no "direction", other than perhaps positive or negative.

Okay, so scalars are what you're used to. In fact we could go so far as to describe the algebra you learned in grade school as *scalar* algebra, or the calculus many of us learned in high school as *scalar* calculus, be cause both dealt almost exclusively with scalars. This is to be contrasted with *vector* calculus or *vector* algebra, that most of us either only got in college if at all. So what *is* a vector?

Definition: A vector is *a list of numbers*. There are (at least) two ways to interpret what this list of numbers mean: One way to think of the vector as being *a point in a space*. Then this list of numbers is a way of identifying that point in space, where each number represents the vector's component that dimension. Another way to think of a vector is *a magnitude and a direction*, e.g. a quantity like velocity ("the fighter jet's velocity is 250 mph north-by-northwest"). In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers.

In this class we'll denote vectors by including a small arrow overtop of the symbol like so: $\vec{a}$. Another common convention you might encounter in other texts and papers is to denote vectors by use of a boldface font ($\boldsymbol{a}$). An example of a vector is $\vec{a} = [4,3]$. Graphically, you can think of this vector as an arrow in the x-y plane, pointing from the origin to the point at x=3, y=4 (see illustration.)

In this example, the list of numbers was only two elements long, but in principle it could be any length. The dimensionality of a vector is the length of the list. So, our example $\vec{a}$ is 2-dimensional because it is a list of two numbers. Not surprisingly all 2-dimentional vectors live in a plane. A 3-dimensional vector would be a list of three numbers, and they live in a 3-D volume. A 27-dimensional vector would be a list of twenty-seven numbers, and would live in a space only Ilana's dad could visualize.



Magnitudes and direction

The "magnitude" of a vector is the distance from the endpoint of the vector to the origin – in a word, it's length. Suppose we want to calculate the magnitude of the vector $\vec{a} = [4,3]$. This vector extends 4 units along the x-axis, and 3 units along the y-axis. To calculate the magnitude $\|\vec{a}\|$ of the vector we can use the Pythagorean theorem ($x^2 + y^2 = z^2$).

$$\|\vec{a}\| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$$

The magnitude of a vector is a scalar value – a number representing the length of the vector independent of the direction. There are a lot of examples were the magnitudes of vectors are important to us: velocities are vectors, speeds are their magnitudes; displacements are vectors, distances are their magnitudes.

To complement the magnitude, which represents the length independent of direction, one might wish for a way of representing the direction of a vector independent of its length. For this purpose, we use "unit vectors," which are quite simply vectors with a magnitude of 1. A unit vector is denoted by a small "carrot" or "hat" above the symbol. For example, $\hat{a}$ represents the unit vector associated with the vector $\vec{a}$. To calculate the unit vector associated with a particular vector, we take the original vector and divide it by its magnitude. In mathematical terms, this process is written as:

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

Definition: A unit vector is a vector of magnitude 1. Unit vectors can be used to express the direction of a vector independent of its magnitude.

Returning to the previous example of $\vec{a} = [4,3]$, recall $\|\vec{a}\| = 5$. When dividing a vector ($\vec{a}$) by a scalar ($\|\vec{a}\|$), we divide each component of the vector individually by the scalar. In the same way, when multiplying a vector by a scalar we will proceed component by component. Note that this will be very different when multiplying a vector by another vector, as discussed below. But for now, in the case of dividing a vector by a scalar we arrive at:

$$\hat{a} = \frac{[4,3]}{5}$$

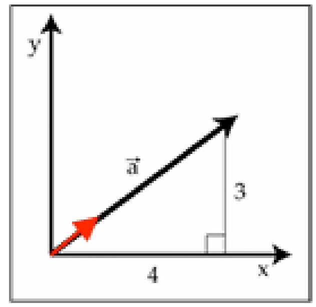
$$\hat{a} = \left[\frac{4}{5}, \frac{3}{5}\right]$$

$$\|\hat{a}\|^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2$$

$$\|\hat{a}\|^2 = \left(\frac{16}{25}\right) + \left(\frac{9}{25}\right)$$

$$\|\hat{a}\|^2 = \left(\frac{25}{25}\right) = 1$$

$$\|\hat{a}\| = 1$$



So we have demonstrated how to create a unit vector $\hat{a}$ that has a magnitude of 1 but a direction identical to the vector $\vec{a}$. Taking together the magnitude $\|\vec{a}\|$ and the unit vector $\hat{a}$ we have all of the information contained in the vector $\vec{a}$, but neatly separated into its magnitude and direction components. We can use these two components to re-create the vector $\vec{a}$ by multiplying the vector $\hat{a}$ by the scalar $\|\vec{a}\|$ like so:

$$\vec{a} = \hat{a} * \|\vec{a}\|$$