



## **Introduction:**

In mathematics, a matrix is also known as matrices. It is a rectangular array of numbers, figures, or expressions, organized in rows and columns. Matrices are usually written in box brackets. In matrices, the horizontal and vertical lines of entries are rows and columns. The size of a matrix is determined by the number of rows and columns that it holds. A matrix with  $m$  rows and  $n$  columns is named an  $m \times n$  matrix or M-by-N matrix, while  $m$  and  $n$  are described its dimensions. The dimensions of the resulting matrix are  $2 \times 3$  up (read “two by three”) as there are 2 rows and 3 columns.

the individual parts that are the numbers, symbols, or expressions in a matrix are named as their entries.

## **Definition:**

- A matrix is a set of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

## **Matrix algebra has at least two advantages:**

- Reduces complicated systems of equations to simple expressions
- Adaptable to systematic method of mathematical treatment and well suited to computers

## **Properties:**

1. Location: the first index represent row  $C_i$  and the second index represent the column  $C_j$

e.g. ( $a_{21}$  is the element in row 2. column 1 of the matrix A)

2. Value: which represent the amount of the element.

- **Examples:**

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

- 3x3 matrix



- 2x4 matrix  $\begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix}$
- 1x2 matrix  $[1 \quad -1]$

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters.

A general real matrix  $A \in R$  with  $m \times n$  elements is of the form:

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

### **Type of Matrices:**

#### **1. Column matrix:**

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

#### **2. Row matrix:**

Only number of columns but only one row

$$[1 \quad 1 \quad 6] \quad [0 \quad 3 \quad 5 \quad 2] \quad [a_{11} \quad a_{12} \quad a_{13} \cdots a_{1n}]$$

#### **3. Rectangular matrix:**

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix} \quad m \neq n$$



#### 4. **Square matrix:**

The number of rows is equal to the number of columns

(a square matrix  $A$  has an order of  $m$ )

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements  $a_{ij}$  for which  $i=j$

#### 5. **Unit or Identity matrix - I**

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e.  $a_{ij}=0$  for all  $i \neq j$

$a_{ij} = 1$  for some or all  $i = j$

#### 6. **Null (zero) matrix - 0**

All elements in the matrix are zero

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a_{ij}=0$  For all  $i, j$

#### 7. **Triangular matrix**

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$



### 7a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e.  $a_{ij} = 0$  for all  $i > j$

### 7b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e.  $a_{ij} = 0$  for all  $i < j$

### 8. Diagonal matrix

Only has non-zero elements on the main diagonal, those non-zero elements can have any value.

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e.  $a_{ij} = 0$  for all  $i \neq j$        $a_{ii} = a$  for all  $i = j$

## Matrices – Operations

### 1. Equality of Matrices

Two matrices are said to be equal only when all corresponding elements are equal

Therefore, their size or dimensions are equal as well

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad A=B$$



Some properties of equality:

- If  $A = B$ , then  $B = A$  for all  $A$  and  $B$
- If  $A = B$ , and  $B = C$ , then  $A = C$  for all  $A$ ,  $B$  and  $C$

$$\bullet \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

- If  $A = B$  then  $a_{ij} = b_{ij}$

## 2. Addition and Subtraction of Matrices

- The sum or difference of two matrices,  $A$  and  $B$  of the same size yields a matrix  $C$  of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Commutative Law:

$$A + B = B + A$$

Associative Law:

$$A + (B + C) = (A + B) + C = A + B + C$$

$$\begin{array}{ccc} \begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} & + & \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \\ A & & B \quad C \\ 2 \times 3 & & 2 \times 3 \quad 2 \times 3 \end{array}$$

$$A + 0 = 0 + A = A$$

$A + (-A) = 0$  (where  $-A$  is the matrix composed of  $-a_{ij}$  as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$



### 3. Scalar Multiplication of Matrices

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then  $kA = Ak$

**Example:** If  $k=4$  and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

#### Properties:

$$k(A + B) = kA + kB$$

$$(k + g)A = kA + gA$$

$$k(AB) = (kA)B = A(k)B$$