4. Multiplication of Matrices

The product of two matrices is another matrix

Two matrices A and B must be conformable for multiplication to be possible

i.e. the number of columns of A must equal the number of rows of B

Example 1:

A	x	B =	C
(1x3)		(3x1)	(1x1)
A	x	B =	Not possible!
(6x2)		(6x3)	

Example 2:

A x B = C

(2x3) (3x2) (2x2)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row i of A with column j of B – row by column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times4)+(2\times6)+(3\times5) & (1\times8)+(2\times2)+(3\times3) \\ (4\times4)+(2\times6)+(7\times5) & (4\times8)+(2\times2)+(7\times3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Assuming that matrices A, B and C are conformable for the operations indicated, the following are true:

1. A(BC) = (AB)C = ABC - (associative law)
2. A(B+C) = AB + AC - (first distributive law)
3. (A+B)C = AC + BC - (second distributive law)

- **Caution!**
- **AB** not generally equal to **BA**, **BA** may not be conformable
- If **AB** = **0**, neither **A** nor **B** necessarily = **0**
- If **AB** = **AC**, **B** not necessarily = **C**

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

- If AB = 0, neither A nor B necessarily = 0

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **<u>Transpose of A Matrix</u>**

If:

$$\underset{2x3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$a_{ij} = a^T_{ji}$ For all i and j

To transpose:

Interchange rows and columns

The dimensions of A^T are the reverse of the dimensions of A

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix} \qquad 2 \text{ x } 3$$

$$A^T =_3 A^{T_2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix} \qquad 3 \times 2$$

- **<u>Properties of transposed matrices:</u>**

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^T = A$

1. $(A+B)^T = A^T + B^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

2. $(AB)^T = B^T A^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

- **<u>Determinant of A Matrix</u>**

1. The determine of a 2 x 2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is written det.

A or $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (−) (+)

$= a_{11} * a_{22} - a_{12} * a_{21}$

Example 1:

Find the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & -7 \end{bmatrix}$

|A|= (1*-7) - (2*4) = -15

2. The determine of a 3 x 3 matrix is written as:

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

Find det. A if $A = \begin{bmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix}$

Solution:

$$|A| = 3 \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} - (-5) \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} + 3 \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

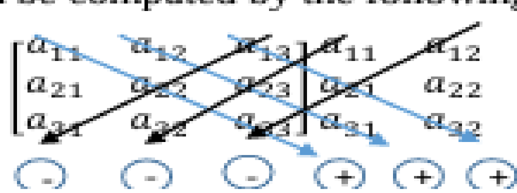
$$|A| = 3(1 * 4 - (-1 * 0)) - (-5)(2 * 4 - (-1 * 1)) + 3(2 * 0 - (1 * 1))$$

$$12 + 45 - 3 = 54$$

There is another to compute the determinant of a 3 x 3 matrix. It is named Sarrus' rule or Sarrus' Scheme.

Consider a 3 x 3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then it is det. Can be computed by the following Scheme:



$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{22}a_{32} - a_{12}a_{21}a_{33}$$

Example 3:

$$|A| = \begin{bmatrix} 3 & -5 & 3 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix} \begin{matrix} 3 & -5 \\ 2 & 1 \\ 1 & 0 \end{matrix}$$

$$= 3*1*4 + (-5*-1*1) + 3*2*0 - (3*1*1) - (3*-1*0) - (-5*2*4) = 54$$

- **Cofactors**

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number i and column j is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

- **Adjoint Matrices**

A cofactor matrix C of a matrix A is the square matrix of the same order as A in which each element a_{ij} is replaced by its cofactor c_{ij}.

$$(adj\ A) = [Cof(A)]^T$$

Example:

Compute the $adjA$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$C_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$A = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$adjA = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$