Derivative of Transcendental Functions or Non-Algebraic Functions :-

مشتقة الدوال المتسامية / غير الجبرية

① Derivative of Trigonometric Functions

مشتقة الدوال المثلثية

Before we get into the derivatives of the trigonometric funs, we need to give a couple of limits that will show up in the derivation

Facts :-

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Examples :-

1- $\lim_{\theta \to 0} \frac{\sin\theta}{6\theta} \Longrightarrow \lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = \frac{1}{6} \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$

$= \frac{1}{6} \times 1 = \boxed{\frac{1}{6}}$ Ans

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2- $\lim_{x\to 0} \frac{\sin(6x)}{x} \longrightarrow \lim_{x\to 0} \frac{\sin(6x)}{x} * \frac{6}{6}$

$= \lim_{x\to 0} \frac{\sin(6x)}{6x} * \frac{6}{1}$

$= 6 \lim_{x\to 0} \frac{\sin 6x}{6x} = 6*1 = \boxed{6}$ Ans

3- $\lim_{x\to 0} \frac{x}{\sin(7x)} \longrightarrow \lim_{x\to 0} \frac{1}{\frac{\sin(7x)}{x}}$

$= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \frac{\sin(7x)}{x} * \frac{7}{7}}$

$= \frac{1}{7 \lim_{x\to 0} \frac{\sin(7x)}{7x}} = \frac{1}{7*1}$

$= \boxed{\frac{1}{7}}$ Ans

4- $\lim_{t\to 0} \frac{\sin(3t)}{\sin(8t)} \longrightarrow \lim_{t\to 0} \frac{\sin(3t)}{\sin(8t)} * \frac{t}{t}$

$= \lim_{t\to 0} \frac{t\sin(3t)}{t\sin(8t)} = \lim_{t\to 0} \frac{\sin(3t)}{t} * \frac{t}{\sin(8t)}$

$= \left(\lim_{t\to 0} \frac{\sin(3t)}{t}\right) * \left(\lim_{t\to 0} \frac{t}{\sin(8t)}\right)$

$= \left(\lim_{t\to 0} \frac{\sin(3t)}{t} * \frac{3}{3}\right) * \left(\lim_{t\to 0} \frac{t}{\sin(8t)} * \frac{8}{8}\right)$

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$$= \left(3 \lim_{t \to 0} \frac{\sin(3t)}{3t}\right) * \left(\frac{1}{8} \lim_{t \to 0} \frac{8t}{\sin(8t)}\right)$$

$$= 3 * \frac{1}{8} = \boxed{\frac{3}{8}} \quad \text{Ans}$$

5- $\lim_{x \to 4} \frac{\sin(x-4)}{x-4}$

let $\theta = x - 4$ & as $x \to 4 \Rightarrow \theta \to 0$

$$\lim_{x \to 4} \frac{\sin(x-4)}{x-4} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \boxed{1} \quad \text{Ans}$$

6- $\lim_{x \to 0} \frac{\cos(2x)-1}{x} \Rightarrow \lim_{x \to 0} \frac{\cos(2x)-1}{x} * \frac{2}{2}$

$$= 2 \lim_{x \to 0} \frac{\cos(2x)-1}{2x} = 2 * 0 = \boxed{0} \quad \text{Ans}$$

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① Derivatives of the Six Trig Fns.

مشتقات الدوال المثلثية الستة

1- $\frac{d}{dx}\sin u = \cos u * u'$

2- $\frac{d}{dx}\cos u = -\sin u * u'$

3- $\frac{d}{dx}\tan u = \sec^2 u * u'$

4- $\frac{d}{dx}\sec u = \sec u \tan u * u'$

5- $\frac{d}{dx}\csc u = -\csc u \cot u * u'$

6 $\frac{d}{dx}\cot u = -\csc^2 u * u'$

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Examples :-

① Show that $\frac{d}{dx}\sin x = \cos x$?

Solution

To prove this derivative, we need to use the derivative by definition formula

$$\frac{dF}{dx} = \lim_{x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \quad \text{—— ①}$$

let $F(x) = \sin x$ —— ②

& $F(x+\Delta x) = \sin(x+\Delta x)$ —— ③

where; $\sin(x+\Delta x) = \sin x \cos\Delta x + \cos x \sin\Delta x$ —— ④

put eq=s ② & ④ into ①, yields;

$$\frac{d}{dx}\sin x = \frac{\sin x\cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos\Delta x - 1) + \cos x \sin\Delta x}{\Delta x}$$

$$= \sin x \overbrace{\lim_{\Delta x \to 0} \frac{\cos\Delta x - 1}{\Delta x}}^{0} + \cos x \overbrace{\lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x}}^{1}$$

$\therefore \frac{d}{dx}\sin x = \boxed{\cos x}$ Ans

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To prove this derivative, we need to use the derivative by definition formula

$$\frac{dF}{dx} = \lim_{x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \quad —①$$

let $F(x) = \sin x \quad —②$

& $F(x+\Delta x) = \sin(x+\Delta x) \quad —③$

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put eq=s ② & ④ into ①, yields;

$$\frac{d}{dx}\sin x = \frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos\Delta x - 1) + \cos x \sin\Delta x}{\Delta x}$$

$$= \sin x \lim_{\Delta x \to 0} \cancelto{0}{\frac{\cos\Delta x - 1}{\Delta x}} + \cos x \lim_{\Delta x \to 0} \cancelto{1}{\frac{\sin\Delta x}{\Delta x}}$$

$\therefore \frac{d}{dx}\sin x = \boxed{\cos x}$ Ans

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$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$\because \cos^2 x + \sin^2 x = 1$

$\therefore \frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$ Ans

④ Derive $g(x) = 3\sec x - 10\cot x$

Solution

$$\frac{d\,g(x)}{dx} = g'(x) = 3\sec x\tan x - 10(-\csc^2 x)$$

$$= \boxed{3\sec x\tan x + 10\csc^2 x}$$ Ans

⑤ Derive $h(w) = 3w^{-4} - w^2\tan w$

Solution

$$\frac{d\,h(w)}{dw} = h'(w) = 3*(-4)w^{-5} - [w^2 * \sec^2 w + \tan w * 2w]$$

$$= \boxed{-12w^{-5} - w^2\sec^2 w - 2w\tan w}$$ Ans

⑥ Derive $y = 5\sin x \cos x + 4\sec x$

Solution

$$\frac{dy}{dx} = y' = 5\left[\sin x * (-\sin x) + \cos x * \cos x\right] + 4\sec x \tan x$$

$$= \boxed{5(\cos^2 x - \sin^2 x) + 4\sec x \tan x}, \text{ Ans}$$

⑦ $p(t) = \dfrac{\sin t}{3 - 2\cos t}$, derive it?

Solution

$$\frac{dp}{dt} = p' = \frac{(3-2\cos t)\cos t - \sin t * (-2(-\sin t))}{(3-2\cos t)^2}$$

$$= \frac{3\cos t - 2\cos^2 t - 2\sin^2 t}{(3-2\cos t)^2}$$

$$= \frac{3\cos t - 2(\overbrace{\cos^2 t + \sin^2 t}^{1})}{(3-2\cos t)^2}$$

$$= \boxed{\frac{3\cos t - 2}{(3-2\cos t)^2}} \quad \text{Ans}$$

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② Derivative of Inverse Trigonometric Functions :- مشتقة معكوس الدوال المثلثية

1- $\dfrac{d}{dx}\sin^{-1}u = \dfrac{u'}{\sqrt{1-u^2}}$

2- $\dfrac{d}{dx}\cos^{-1}u = \dfrac{-u'}{\sqrt{1-u^2}}$

3- $\dfrac{d}{dx}\tan^{-1}u = \dfrac{u'}{1+u^2}$

4- $\dfrac{d}{dx}\sec^{-1}u = \dfrac{u'}{|u|\sqrt{u^2-1}}$

5- $\dfrac{d}{dx}\csc^{-1}u = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

6- $\dfrac{d}{dx}\cot^{-1}u = \dfrac{-u'}{1+u^2}$

Fact :- حقيقة

IF $f(x)$ & $g(x)$ are inverses of each other then,

$$g'(x) = \frac{1}{f'(g(x))}$$

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Examples :-

① Show that $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$?

Solution)

By previous fact;

let $f(x) = \sin x \Rightarrow f'(x) = \cos x$

& $g(x) = \sin^{-1}x$

Then, $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1}x)}$

$\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\cos(\sin^{-1}x)}$

To Find the denominator, we do the follows; لإيجاد مقام المعادلة نقوم بعمل التالي

If $y = \sin^{-1}x \Longrightarrow \therefore x = \sin y$

$\therefore \cos(\sin^{-1}x) = \cos y$

But $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \sqrt{1-\sin^2 y}$

$\therefore \cos(\sin^{-1}x) = \sqrt{1-\sin y} = \sqrt{1-x^2}$

$\boxed{\therefore \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$ Ans

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② Show that $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$?

Solution

By using the fact,

let $f(x) = \cos x \Longrightarrow f'(x) = -\sin x$

& $g(x) = \cos^{-1}x$

Then $g'(x) = \frac{d}{dx}\cos^{-1}x = \frac{1}{f'(g(x))} = \frac{1}{-\sin(\cos^{-1}x)}$

$\therefore \frac{d}{dx}\cos^{-1}x = \frac{1}{-\sin(\cos^{-1}x)} = \frac{-1}{\sin(\cos^{-1}x)}$

To find the equation's denominator, we do the follows; لايجاد مقام المعادلة نعمل التالي

If $y = \cos^{-1}x \Longrightarrow \therefore x = \cos y$

& $\sin(\cos^{-1}x) = \sin y$

But $\cos^2 y + \sin^2 y = 1 \Longrightarrow \sin y = \sqrt{1-\cos^2 y}$

$\therefore \sin(\cos^{-1}x) = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

$\therefore \boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}}$ Ans

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③ Show that $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$?

Solution)

By using the fact

let $f(x) = \tan x \Longrightarrow f'(x) = \sec^2 x$

& $g(x) = \tan^{-1}x$

Then $g'(x) = \frac{1}{f'(g(x))} \Longrightarrow \frac{d}{dx}\tan^{-1}x = \frac{1}{\sec^2(\tan^{-1}x)}$

To Find the denominator, we do the follows;

لإيجاد مقام العلاقة اعلاه نعمل التالي :

If $y = \tan^{-1}x \Longrightarrow \therefore x = \tan y$

$\therefore \sec^2(\tan^{-1}x) = \sec^2 y$

But $\{\cos^2 y + \sin^2 y = 1\} \div \cos^2 y$

$\Longrightarrow 1 + \tan^2 y = \sec^2 y$

$\therefore \sec^2(\tan^{-1}x) = 1 + \tan^2 y = 1 + x^2$

$\therefore \boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}}$ Ans

④ Derive $f(t) = 4\cos^{-1}t - 10\tan^{-1}t$?

Solution

$$\frac{d\,f(t)}{dt} = \boxed{\frac{-4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}} \quad \text{Ans}$$

⑤ Derive $y = \sqrt{z}\,\sin^{-1}z$?

Solution

هنا لدينا دالتين مضروبات ببعض وهما $\sqrt{z}$, $\sin^{-1}z$
ومشتقة الدالة y هي مشتقة حاصل ضرب ما بين
الدالتين ، وكما يلي :

$$\frac{dy}{dz} = y' = \sqrt{z} \times \frac{1}{\sqrt{1-z^2}} + \sin^{-1}z \times \frac{1}{2} z^{-\frac{1}{2}}$$

$$= \frac{\sqrt{z}}{\sqrt{1-z^2}} + \frac{\sin^{-1}z}{2\sqrt{z}}$$

$$= \boxed{\sqrt{\frac{z}{1-z^2}} + \frac{\sin^{-1}z}{2\sqrt{z}}} \quad \text{Ans}$$

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