Al-Mustaqbal University / College of Engineering & Technology
Class: first
Subject: integral Mathematics/Code: UOMU024024
Lecturer: M.Sc. Alaa Khalid
Lecture name: introduction of integration
Lecture: 10
2nd term

2.4 Length of a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \qquad with\ x-axis$$

Or

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy \qquad with\ y-axis$$

If x=f (t) , y =f (t) then

$$L = \int_{t1}^{t2} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\ dt$$

Example 1: Find the length of the curve bounded by the curve $y = \frac{4\sqrt{2}}{3}x^{\frac{3}{2}} - 1 \quad and\ 0 \le x \le 1$.

Solution //

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} * \frac{3}{2}x^{\frac{1}{2}} = 2\sqrt{2}x^{\frac{1}{2}}$$

$$L=\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \quad \rightarrow L = \frac{2}{3} * \frac{1}{8}(1+8x)^{\frac{3}{2}}\Big|_0^1 = \frac{13}{6}$$

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Example/

Let f(x) = 2x $^{3/2}$ Calculate the arc length of the graph of f(x) over the interval [0, 1].

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_0^1 \sqrt{1+9x}\,dx.$$

Substitute $u = 1+9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\text{Arc Length} = \int_0^1 \sqrt{1+9x}\,dx = \frac{1}{9}\int_0^1 \sqrt{1+9x}\,9dx = \frac{1}{9}\int_0^1 \sqrt{u}\,du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \longrightarrow \approx 2.268 \text{ units.}$$

Ex / Find the **arc length** of the curve $y = x^{3/2}$ from 1 to 2

Method 1 :

1) $y = x^{3/2}$ from x=1 to x= 2

$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ is define on [1,2]

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \rightarrow L = \int_1^2 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_1^2 \frac{9}{4}\left(1+\frac{9}{4}x\right)^{1/2}dx$$

$$= \frac{4}{9}\,\frac{\left[1+\frac{9}{4}x\right]^{3/2}}{3/2}\Big]_1^2 = \frac{4}{9}\,\frac{2}{3}\left[\left(1+\frac{9}{4}x\right)^{3/2}\right]_1^2 = \frac{8}{27}\left[\left(1+\frac{9}{4}(2)\right)^{3/2} - \left(1+\frac{9}{4}(1)\right)^{3/2}\right]$$

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3- if $y = f(t)$ and $x = g(t)$ then $L = \int_{t1}^{t2} \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}\ dt$

Ex / Find the circumference of acircle of radius a .

equation of circle of radius a is $x^2 + y^2 = a^2$

and the parametric equations are

$x = a\cos t \quad , \quad y = a\sin t$

$\Downarrow \qquad\qquad \Downarrow$

$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = a\cos t$

$$L = \int_{t1}^{t2} \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}\ dt$$

$$L = \int_0^{2\pi} \sqrt{(a\cos t)^2 + (-a\sin t)^2}\ dt$$

$$= \int_0^{2\pi} \sqrt{a^2(\cos^2 t + \sin^2 t)}\ dt = \int_0^{2\pi} a\, dt = a[t]_0^{2\pi} = 2\pi a$$

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