Lecture Three

Methods of analysis

3.1Introduction

Having understood the fundamental laws of circuit theory (Ohm's lawand Kirchhoff's laws), we are now prepared to apply these laws to develop two powerful techniques for circuit analysis: nodal analysis, which is based on a systematic application of Kirchhoff's current law (KCL), and mesh analysis, which is based on a systematic application of Kirchhoff's voltage law (KVL). With the two techniques to be developed, we can analyze almost any circuit by obtaining a set of simultaneous equations that are then solved to get the required current or voltage values.

3.2 Nodal analysis

Nodal analysis provides a general procedure for analyzing circuits using node voltages as the circuit variables. Choosing node voltages instead of element voltages as circuit variables is convenient and reduces the number of equations one must solve simultaneously. To simplify matters, we shall assume in this section that circuits do not contain voltage sources. Circuits that contain voltage sources will be analyzed in the next section.

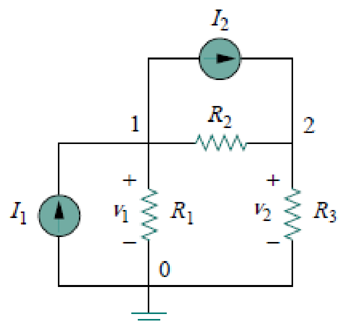
Steps to Determine Node Voltages:

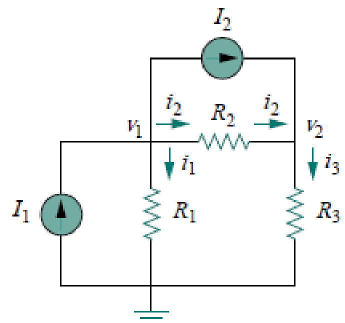
1. Select a node as the reference node. Assign voltages $v_1, v_2, . . v_{n-1}$ to the remaining **n-1** nodes. The voltages are referenced with respect to the reference node.
2. Apply **KCL** to each of the **n-1** nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

Once a reference node is selected, voltages are assigned to nonreference nodes relative to it. For example, in Fig. 3.2(a), node 0 is the reference (v = 0), while nodes 1 and 2 have voltages v_1 and v_2. These voltages represent the potential rise from the reference node. Next, we apply Kirchhoff's Current Law (KCL) to each nonreference node. To simplify visualization, Fig. 3.2(b) introduces currents i_1, i_2, and i_3 through resistors R_1, R_2, and R_3, respectively. Applying KCL at node 1 gives:

At node 1: $\boldsymbol{I_1 = I_2 + i_1 + i_2}$ (3.1)

At node 2: $\boldsymbol{I_2 + i_2 = i_3}$ (3.2)





We now apply Ohm's law to express the unknown currents i_1, i_2, and i_3 in terms of node voltages.

Current flows from a **higher** potential to a **lower** potential in a resistor

We can express this principle as $$i = \frac{v_{higher} - v_{lower}}{R} \quad (3.3)$$

With this in mind, we obtained:

$$i_1 = \frac{v_1 - 0}{R_1}, \quad \text{or } i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2}, \quad \text{or } i_2 = G_2 (v_1 - v_2) \quad (3.4)$$

$$i_3 = \frac{v_2 - 0}{R_3}, \quad \text{or } i_3 = G_3 v_2$$

Substituting **Eq. (3.4) in Eqs. (3.1)** and **(3.2)** results, respectively, in

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} \tag{3.5}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3} \tag{3.6}$$

In terms of the conductance. Eqs. (3.5) and (3.6) become

The conductance *G* of a resistor is the reciprocal of its resistance

$$G = \frac{1}{R}$$

$$I_1 = I_2 + G_1 v_1 + G_2 (v_1 - v_2) \tag{3.7}$$

$$I_1 - I_2 = G_1 v_1 + G_2 v_1 - G_2 v_2$$

$$I_1 - I_2 = (G_1 + G_2) v_1 - G_2 v_2$$

$$I_2 + G_2 (v_1 - v_2) = G_3 v_2 \tag{3.8}$$

$$I_2 = G_3 v_2 - G_2 (v_1 - v_2)$$

$$I_2 = G_3 v_2 - G_2 v_1 + G_2 v_2$$

$$I_2 = - G_2 v_1 + G_2 v_2 + G_3 v_2$$

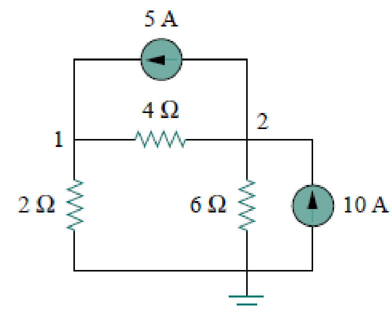
$$I_2 = - G_2 v_1 + (G_2 + G_3) v_2$$

To obtain the node voltages v_1 and v_2 using any standard method, such as the elimination method or Cramer's rule.

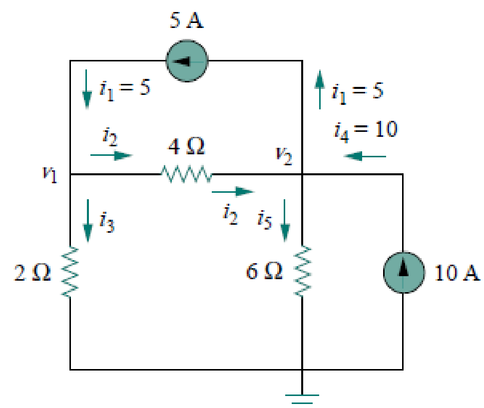
To use Cramer's rule, one must cast the simultaneous equations in matrix form. For example, Eqs. (3.7) and (3.8) can be cast in matrix form as

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

Example (1): *Calculate the node voltages in the circuit shown in the figure below*



Solution



Assign nodes 1 and 2 to determine voltages v_1 and v_2.

Selected the current to apply KCL

At node 1, applying KCL and Ohm's law gives

$i_1 = i_2 + i_3 \;\Rightarrow 5 = \frac{\mathbf{v_1 - v_2}}{\mathbf{4}} + \frac{\mathbf{v_1 - 0}}{\mathbf{2}}$

Multiplying each term in the last equation by 4, we obtain

$20 = v_1 - v_2 + 2v_1 \quad \Rightarrow \; \boldsymbol{3v_1 - v_2 = 20}$ eq (1)

At node 2, we do the same thing and get

$i_2 + i_4 = i_1 + i_5 \;\Rightarrow \frac{\mathbf{v_1 - v_2}}{\mathbf{4}} + 10 = 5 + \frac{\mathbf{v_2 - 0}}{\mathbf{6}}$

Multiplying each term by 12 results in

$3v_1 - 3v_2 + 120 = 60 + 2v_2 \quad \Rightarrow \; \boldsymbol{-3v_1 + 5v_2 = 60}$ eq (2)

Method 1 *Using the elimination technique, from eq(1&2)*

$\boldsymbol{-3v_1 + 5(3v_1 - 20) = 60 \Rightarrow \; -3v_1 + 15v_1 - 100 = 60 \;\Rightarrow\; 12v_1 = 160}$

$\Rightarrow \boldsymbol{v_1 = 13.33V}$

$v_2 = 3v_1 - 20 \Rightarrow \quad v_2 = 3(13.33) - 20 = 20V$

Method 2 To use Cramer's rule, we need to put Eqs. (1) and (2) in matrix form as

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

The Cramer's rule: $v_1 = \frac{\Delta_1}{\Delta}$, $\quad v_2 = \frac{\Delta_2}{\Delta}$

The determinant of the matrix is:

$$\Delta = \begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} = 15 - 3 = 12$$

We now obtain v_1 and v_2 as

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{bmatrix} 20 & -1 \\ 60 & 5 \end{bmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333\text{V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{bmatrix} 3 & 20 \\ -3 & 60 \end{bmatrix}}{\Delta} = \frac{180 + 60}{12} = 20\text{V}$$

giving us the same result as the elimination method.

If we need the currents, we can easily calculate them from the values of the nodal voltages.

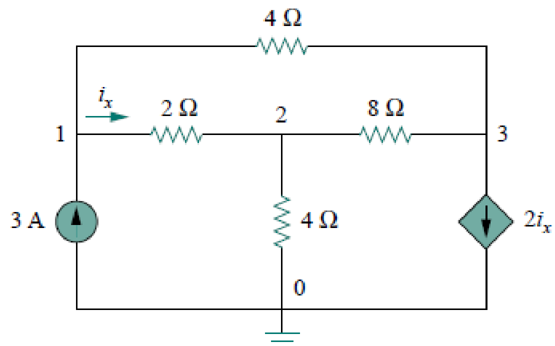
$i_1 = 5$ A, $\quad i_2 = \frac{\mathbf{v_1 - v_2}}{\mathbf{4}} = \frac{\mathbf{13.33 - 20}}{\mathbf{4}} = -1.6667$ A,

$i_3 = \frac{\mathbf{v_1}}{\mathbf{2}} = 6.666$A $\quad i_4 = 10$ A,

$i_5 = \frac{\mathbf{v_2}}{\mathbf{6}} = 3.333$ A

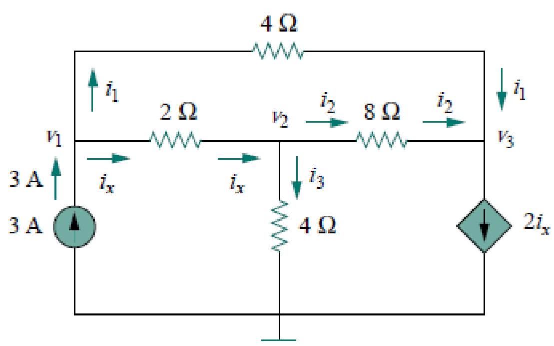
The fact that i_2 is negative shows that the current flows in the direction opposite to the one assumed

Example (2): Determine the voltages at the nodes in the figure below



Solution

The circuit in this example has three nonreference nodes, unlike the previous example which has two nonreference nodes. We assign voltages to the three nodes as shown in the figure below and label the currents



At node 1, $3 = i_1 + i_x \quad \Rightarrow \quad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$

Multiplying by **4** and rearranging terms, we get

$$3v_1 - 2v_2 - v_3 = 12 \qquad \text{eq(1)}$$

At node 2,

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

Multiplying by 8 and rearranging terms, we get

$$-4v_1 + 7v_2 - v_3 = 0 \qquad \text{eq(2)}$$

At node 3,

$$i_1 + i_2 = 2i_x \quad \Rightarrow \quad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

Multiplying by 8, rearranging terms, and dividing by 3, we get

$$2v_1 - 3v_2 + v_3 = 0 \qquad \text{eq(3)}$$

We have three simultaneous equations to solve to get the node voltages v_1, $v2$, and $v3$. We shall solve the equations in two ways

METHOD 1 Using the elimination technique, we add Eqs. (1 & 3)

$5v_1 - 5v_2 = 12$

$v_1 - v_2 = \frac{12}{5}$ eq(4)

Adding Eqs. (2) and (3) gives

$-2v_1 + 4v_2 = 0 \Rightarrow v_1 = 2v_2$ eq(5)

Substituting Eq. (5) into Eq. (4) yields

$2v_2 - v_2 = 2.4 \quad \Rightarrow \quad v2 = 2.4$ V, $\quad v1 = 2v_2 = 4.8$ V

From Eq. (3), we get

$v3 = 3v_2 - 2v_1 = 3v_2 - 4v_2 = - v_2 = - 2.4$ V

Thus,

$v_1 = 4.8$ V, $\qquad v_2 = 2.4$ V, $\qquad v_3 = -2.4$ V

METHOD 2 To use Cramer's rule, we put Eqs. (1) to (3) in matrix form.

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

From this, we obtain

$$v_1 = \frac{\Delta_1}{\Delta}, \qquad v_2 = \frac{\Delta_2}{\Delta}, \qquad v_3 = \frac{\Delta_3}{\Delta}$$

where Δ_1, Δ_2, and Δ_3 are the determinants to be calculated as follows

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \\ 3 & -2 & -1 \\ -4 & 7 & -1 \end{vmatrix}$$

$= 21 - 12 + 4 + 14 - 9 - 8 = 10$

Similarly, we obtain

$$\Delta_1 = \begin{vmatrix} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \\ 12 & -2 & -1 \\ 0 & 7 & -1 \end{vmatrix} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \\ 3 & 12 & -1 \\ -4 & 0 & -1 \end{vmatrix} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & 12 \\ -4 & 7 & 0 \end{vmatrix} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

Thus, we find

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8 \text{ V}, \qquad v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4 \text{ V}$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4 \text{ V}$$

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Find the voltages at the three nonreference nodes in the circuit of the figure

Answer: $v_1 = 80$ V, $v_2 = -64$ V, $v_3 = 156$ V.