Al- Mustaqbal University

College of Science

Medical Physics Department

First Stage

Mechanics

Lecture Four: General Motion of a Particle in Three Dimensions

Lecturer: Dr.Mokhalad Ali Al-Absawe

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8 The Del Operator

If the force field is conservative so that the components are given by the partial derivative of potential energy function.

اذا كان مجال القوه محافظاً عندها يمكن لمركبات القوة أن تعطى بدلالة المشتقات الجزئية لدالة الطاقة الكامنة.

We can now express a conservative force F vectorially as:

$$\vec{F} = -i\,\frac{\partial V}{\partial x} - j\,\frac{\partial V}{\partial y} - k\,\frac{\partial V}{\partial z}$$

$$\vec{F} = -\nabla V \;\ldots\ldots (1)$$

Where: $\vec{\nabla}$ is del operator given as:

$$\vec{\nabla} = i\,\frac{\partial}{\partial x} + j\,\frac{\partial}{\partial y} + k\,\frac{\partial}{\partial z} \;\ldots\ldots (2)$$

1. $\vec{\nabla}V = \text{Gradient } V \text{ or } (\text{grad } V)$

- Mathematically, the gradient of a function is a vector that represents the maximum spatial derivative of the function in direction and magnitude.
- Physically, the negative gradient of the potential energy function gives the direction and magnitude of the force that acts on a particle located in a field created by other particles.
- The meaning of the negative sign is that the particle is urged to move in the direction of decreasing potential energy rather than in the opposite direction.

- ∇V يسمى بانحدار V.
- رياضياً يعني التفاضل الموضعي للدالة في المقدار والاتجاه .
- فيزيائياً يعني أن الانحدار السالب لدالة الطاقه الكامنه يعطي اتجاه ومقدار القوة التي تؤثر على جسيم موضوع في مجال ناتج عن جسيمات اخرى.
- الاشارة السالبه تعني ان الجسيم اجبر على الحركة باتجاه تناقص الطاقه الكامنه بدلاً من الاتجاه المعاكس.

2. $\vec{\nabla} \times F = \text{Curl } \vec{f}$

- $\vec{\nabla} \times \vec{F}$ يسمى بدوران (التفاف) متجه القوة $\vec{F}$

The condition that a force be conservative can be written compactly as

$\vec{\nabla} \times \vec{F} = 0$ … … (4) (Then The Force $\vec{F}$ is Conservative)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

3. $\vec{\nabla} \cdot \vec{F} = \text{divergence of } \vec{F}$

$(\vec{\nabla} \cdot \vec{F})$ is called the divergence of $\vec{F}$ which gives a measure of the density of the sources of the field at a given point, which is of particular importance in the theory of electricity and magnetism.

$(\vec{\nabla} . \vec{F})$ يمثل تفرق (تباعد) $\vec{F}$ وهي مقياس لكثافة المجال في نقطه معينه وهي مهمة في النظريه الكهربائية والمغناطيسية.

Example: Find the force field of the potential Function $V = x^2 + xy + xz$.

Solution:

Applying the $\vec{\nabla}$ operator

$$\vec{F} = -\vec{\nabla} V$$
$$= -(2xi + jx + kx)$$
$$= -2xi - jx - kx$$

Example: Is the force field $\vec{F} = ixy + jxz + kyz$ conservative?

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & yz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = i\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xz)\right) - j\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xy)\right) + k\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(xy)\right)$$

$\vec{\nabla} \times \vec{F} = i\,(z - x) - j(0) + k\,(z - x)$

$\therefore \vec{\nabla} \times \vec{F} \neq 0 \quad \rightarrow \ \vec{F}$ is non conservative. النتيجه لا تساوي صفر، بالتالي المجال غير محافظ

Example: For what values of the constants a, b and c is the force

$\vec{F} = i(ax + by^2) + jcxy$ conservative?

Solution:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax + by^2 & cxy & 0 \end{vmatrix}$$

$$= i\,(0 - 0) - j(0) + k\,(cy - 2by)$$

$$= k\,(c - 2b)y$$

For conservative force must $\vec{\nabla} \times \vec{F} = 0$

$\therefore c - 2b = 0$

$c = 2b$

So $\vec{F}$ be conservative when $c = 2b$

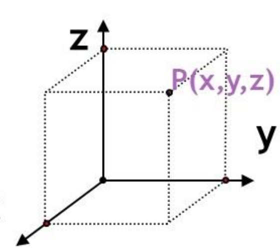
عندما $c = 2b$ تكون $\vec{\nabla} \times \vec{F} = 0$ فاذن $\vec{F}$ عندها تكون محافظة. قيمة a لا أهمية لها.

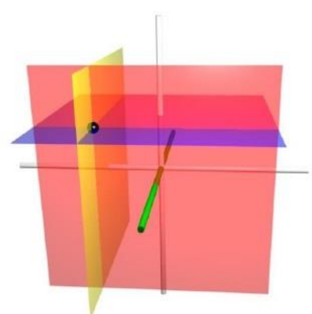
9 The Del Operator in Other Coordinates

هناك عدة انواع من المحاور المستخدمة مثل المحاور الكارتيزية و الاسطوانية و الكروية عليه فان مؤثر $\vec{\nabla}$ يكتب بصور مختلفة اعتمادا على نوع المحاور المستخدمة

1. Cartesian Coordinates (Rectangular Coordinates)

P (x, y, z)

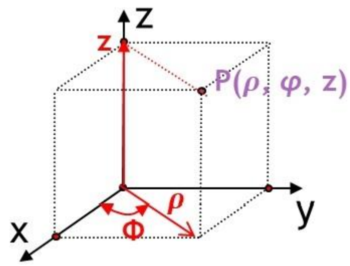


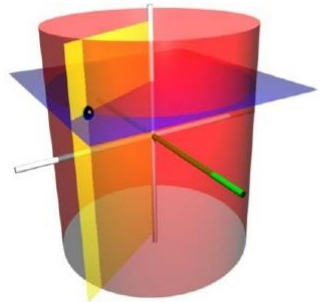


2. Cylindrical Coordinates

P (*r*, *φ*, *z*)

$x = \rho \cos \varphi$
$y = \rho \sin \varphi$
$z = z$

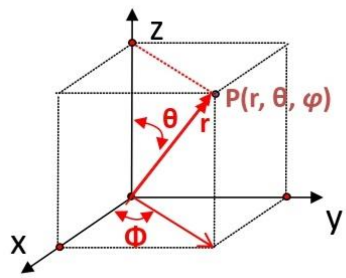


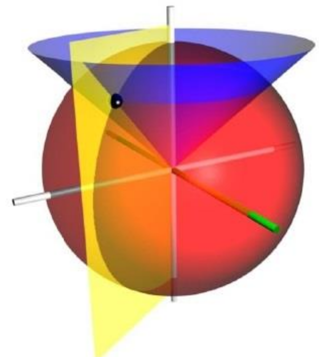


3. Spherical Coordinates

P (*r*, θ, *φ*)

$x = r\sin\theta \cos \varphi$
$y = r\sin\theta \sin \varphi$
$z = r\cos \theta$





الجدول التالي يبين العلاقات بين المحاور في الاحداثيات المختلفة

Conversion between Cartesian, cylindrical, and spherical coordinates

		From		
		Cartesian	**Cylindrical**	**Spherical**
To	**Cartesian**		$x = \rho \cos \varphi$ $y = \rho \sin \varphi$ $z = z$	$x = r\sin\theta \cos \varphi$ $y = r\sin\theta \sin \varphi$ $z = r\cos \theta$
	Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\varphi = \arctan\left(\frac{y}{x}\right)$ $z = z$		$\rho = r \sin \varphi$ $\varphi = \varphi$ $z = r\cos \theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arctan\left(\frac{z}{r}\right)$ $\varphi = \arctan\left(\frac{y}{x}\right)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan\left(\frac{\rho}{z}\right)$ $\varphi = \varphi$	

10 The Harmonic Oscillator in Two and There Dimensions

Consider the motion of a particle subject to a linear restoring force that is always directed toward a fixed point, the origin of our coordinate system. Such a force can be represented by the expression.

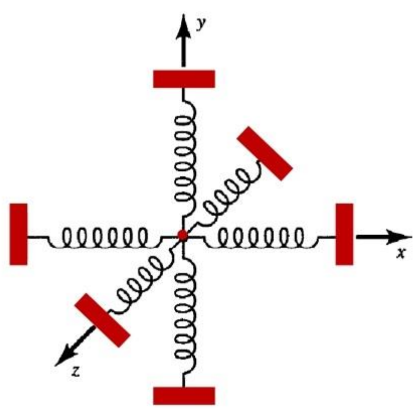
(الحركة الاهتزازيه التوافقيه ببعدين وبثلاثة ابعاد)

لنفرض حركة جسيم تحت تأثير قوه خطيه معيدة لهذا الجسيم الى موضع استقراره بعد ازاحته في هذا الموضع.
وهذه القوة المعيدة تتجه نحو نقطه ثابته وهي نقطه الاصل في نظام الاحداثيات.

$$\vec{F} = -k\vec{r} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \dots\dots (1)$$

The motion of particle in three dimensions represent as particle attached to a set of elastic springs as shown in Figure:

حركة الجسيم بثلاثة ابعاد تمثل بجسيم مربوط بثلاثة نوابض متحدة المركز تمثل نقطة الاصل للاحداثيات الكارتيزية (x,y,z)



A model of a three-dimensional harmonic oscillator

1. Harmonic Motion in Two Dimension

In the case of motion in a single plane (two dimensions), force two component equations equivalent to:

$$\left.\begin{aligned} F_x &= m\ddot{x} = -kx \\ F_y &= m\ddot{y} = -ky \end{aligned}\right\} \ldots\ldots\ldots(2)$$

These are separated, and we can immediately write down the solutions in the form:

المعادلتين (2) تمثلان معادلة متذبذب توافقي في خط مستقيم (حركة اهتزازية مستقلة) احداهما باتجاه (x) والاخرى باتجاه (y) وعلى سطح مستوي واحد هو (xy)

$$x = A\cos(wt + \alpha) \ldots\ldots(3)$$

$$y = B\cos(wt + \beta) \ldots\ldots(4)$$

Where $w = \sqrt{\frac{k}{m}}$

The constants of integration A, B, α, andβ are determined from the initial conditions in any given case.

β, α, B, A ثوابت يمكن ايجادها في الشروط الابتدائيه لاية حالة معطاة.

To find the equation of the path, we eliminate the time t between the two equations.

لايجاد معادلة المسار لحركة هذا الجسيم نحذف الزمن t لكل من هاتين الحركتين

$$y = B\cos(wt + \alpha + \Delta) \ldots\ldots(5)$$

Where: $\Delta = \beta - \alpha$

$$\beta = \alpha + \Delta \ldots\ldots(6)$$

Cosine for sum of two angles is: $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$$\therefore\ \cos(wt + \alpha + \Delta) = \cos(wt + \alpha)\cos\Delta - \sin(wt + \alpha)\sin\Delta \ldots\ldots(7)$$

Eqn. (5) became:

$$y = B\left[\cos(wt + \alpha)\cos\Delta - \sin(wt + \alpha)\sin\Delta\right] \ldots\ldots(8)$$

Eqns. (3) and (4) rewrite as:

$$\frac{x}{A} = \cos(wt + \alpha) \ldots\ldots(9)$$

$$\frac{y}{B} = \cos(wt + \beta) \ldots\ldots(10)$$

From Eqn. (8)

$$\frac{y}{B} = \cos(wt+\alpha)\cos\Delta - \sin(wt+\alpha)\sin\Delta \ \ldots\ldots(11)$$

From. Eqn. (9)

$$\cos(wt+\alpha) = \frac{x}{A}$$

Also,

$$\sin(wt+\alpha) = (1-\cos^2(wt+\alpha))^{1l2}$$

Then Eqn. (11) be:

$$\frac{y}{B} = \frac{x}{A}\cos\Delta - \ (1-\cos^2(wt+\alpha))^{1l2}\sin\Delta$$

$$\frac{y}{B} = \frac{x}{A}\cos\Delta - \left(1-\frac{x^2}{A^2}\right)^{1/2}\sin\Delta$$

$$\frac{y}{B} - \frac{x}{A}\cos\Delta = -\left(1-\frac{x^2}{A^2}\right)^{1/2}\sin\Delta$$

Squaring both sides:

$$\frac{y^2}{B^2} - 2\,\frac{yx}{BA}\cos\Delta + \frac{x^2}{A^2}\cos^2\Delta = \left(1-\frac{x^2}{A^2}\right)\sin^2\Delta$$

$$\frac{y^2}{B^2} - \frac{2yx}{BA}\cos\Delta + \frac{x^2}{A^2}\cos^2\Delta + \frac{x^2}{x^2}\sin^2\Delta = \sin^2\Delta$$

$$\frac{y^2}{B^2} - \frac{2yx}{BA}\cos\Delta + \frac{x^2}{A^2}(\cos^2\Delta + \sin^2\Delta) = \sin^2\Delta$$

$$\frac{x^2}{A^2} - 2\frac{xy}{AB}\cos\Delta + \frac{y^2}{B^2} = \sin^2\Delta \ \ldots\ldots(11)$$ ***Quadratic equation in x and y***

Now the general quadratic

$$ax^2 + bxy + cy^2 + dx + ey = f \ \ldots\ldots(12)$$

Eqn. (12) represents an ***ellipse***, a ***parabola***, or a ***hyperbola***, depending on whether the discriminant (b^2-4ac)

المعادلة (12) تمثل قطع ناقص (ellipse) او قطع مكافئ (Parabola) او قطع زائد (hyperbola) وهذا يعتمد على قيمة او اشارة الحد(b^2-4ac) اي اما سالب او صفر او موجب على التوالي.

a ,b and c are calculated from the comparison between eqn.(11) and Eqn. (12)

$a = \frac{1}{A^2}$, $b = -\frac{2\cos\Delta}{AB}$, $c = \frac{1}{B^2}$

$$\left[\frac{2\cos\Delta}{AB}\right]^2 - 4\frac{1}{A^2B^2} = b^2 - 4ac$$

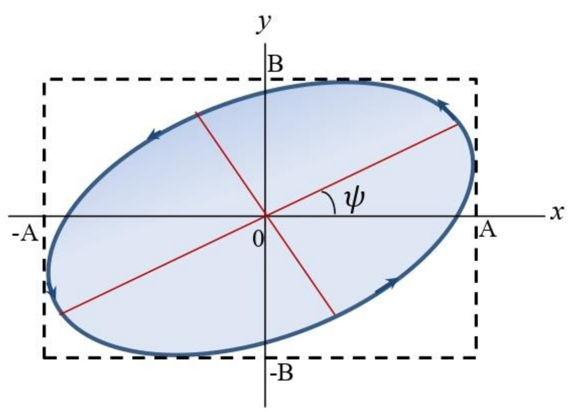
$$= \frac{4\cos^2\Delta}{A^2B^2} - 4\frac{1}{A^2B^2}$$

$$= \frac{4\,(\cos^2\Delta - 1)}{A^2B^2} = \frac{-4\sin^2\Delta}{A^2B^2}$$

$$= -\left(\frac{2\sin\Delta}{AB}\right)^2$$

In our case the discriminant is equal $(-\left(\frac{2\sin\Delta}{AB}\right)^2)$, which is negative, so the path is an ellipse as shown in Figure

هنا الحد المميز مقدار سالب لذا نحصل على شكل بيضوي.



- If the phase difference

$\Delta = \frac{\pi}{2}$

The Eqn. of path:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

Which is Eqn. of ***ellipse*** whose axes coincide with the coordinate axes.

هنا مسار الحركة يمثل معادلة قطع ناقص محاوره تتطابق مع المحاور الكارتيزية

$\Delta = 0 \;\; or\; \pi$

Then the Eqn. of path reduces to that of ***straight line***:

$y = \pm \frac{B}{A} x$ هنا مسار الحركة يكون بشكل خط مستقيم

Where (-) negative for $\Delta = 0$

(+) positive for $\Delta = \pi$

- In the general case, it is possible to show that the axis of the ***elliptical path*** is inclined to the x-axis by the angle (ψ) where

$\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2}$

بشكل عام ممكن نحصل على مسار بيضوي يميل عن المحاور بزاوية مقدارها (ψ)

Example:

Find the potential energy for harmonic oscillator: a) two dimensions b) three dimensions

Solution:

a. Two Dimensions

$\vec{F} = -k_1 x_i - k_2 y j$

$\because \vec{\nabla} \times \vec{F} = 0$ Conservative force

.: مجال القوة محافظ اذن يوجد جهد V يحقق العلاقه $\vec{F} = -\nabla V$

$$F = -k_1 x_i - k_2 y j = -\nabla V = -\frac{dV}{dx} i - \frac{dV}{dy} j - \frac{dV}{dz} k$$

$$\therefore \frac{dV}{dx} = k_1 x \rightarrow dV = k_1 x\, dx = \frac{1}{2} k_1 x^2$$

$$\frac{dV}{dy} = k_1 y \rightarrow dV = k_1 y\, dy = \frac{1}{2} k_1 y^2$$

b. Three dimensions

$$F = -k_1 x i - k_2 y j - k_3 z k$$

Also, $\vec{\nabla} \times \vec{F} = 0$ Conservative force

$$\therefore \frac{dV}{dx} = k_1 x \rightarrow V_x = \frac{1}{2} k_1 x^2$$

$$\frac{dV}{dy} = k_2 y \rightarrow V_y = \frac{1}{2} k_2 y^2$$

$$\frac{dV}{dz} = k_2 z \rightarrow V_z = \frac{1}{2} k_2 z^2$$

$$\therefore V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2 + \frac{1}{2} k_3 z^2$$