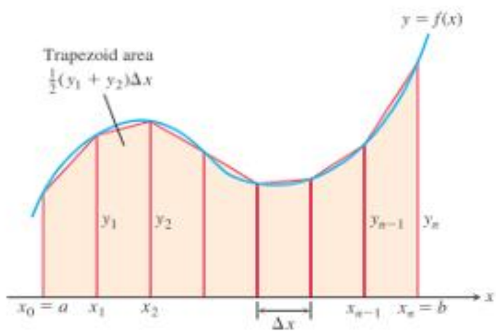
Numerical Integration

Trapezoidal Approximation

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in the figure below. It is not necessary for the subdivision points $x_0, x_1, x_2, \dots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. Therefore, we assume that the length of each subinterval is:

$$\Delta x = \frac{b-a}{n}$$



The area of the trapezoid that lies above the ith subinterval is:

$$A = \frac{\Delta x (y_{i-1} + y_i)}{2}$$

The Trapezoidal Rule

To approximate $\int_a^b f(x)dx$, use:

$$A = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

Example 1: Use the Trapezoidal Rule with n = 4 to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

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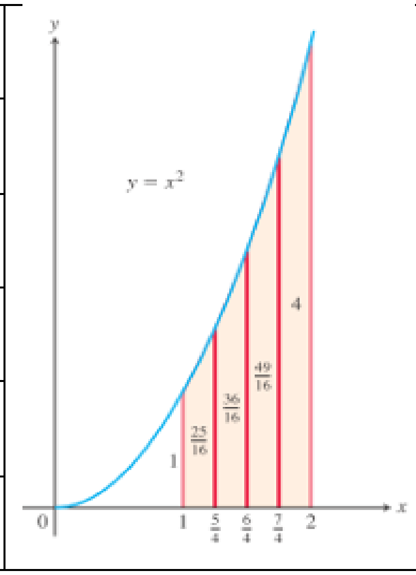
Example 1: Use the Trapezoidal Rule with n = 4 to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution:

$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

x	$y = x^2$
1	1
5/4	25/16
6/4	36/16
7/4	49/16
2	4



$$A = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) = \frac{75}{32} = 2.34375$$

The exact value of the integral is:

$$\int_1^2 x^2 dx = \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.33333$$

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Simpson's Rule: Approximations Using Parabolas

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (the figure below), where $h = \Delta x = (b - a) / n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form:

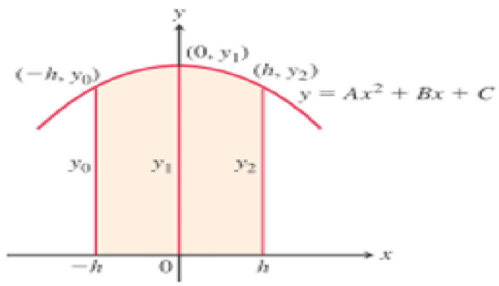
$$y = Ax^2 + Bx + C$$

As a result, the area under it from x= -h to x=h is:

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx$$

$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \Big]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C)$$



Also, we have:

$$y_0 = Ah^2 - Bh + C$$

$$y_1 = C$$

$$y_2 = Ah^2 + Bh + C$$

From which, we obtain:

$$C = y_1$$

$$Ah^2 - Bh = y_0 - y_1$$

$$Ah^2 + Bh = y_2 - y_1$$

$$2Ah^2 = y_0 + y_2 - 2y_1$$

Substitute these equations into A_p, we have:

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + y_2 - 2y_1 + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Computing the areas under all the parabolas and adding the results gives the approximation:

$$\int_a^b f(x)\, dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

$$S = \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) = 32\frac{1}{12}$$

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The Simpson's Rule

$$S = \int_a^b f(x)\,dx = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

The number n is **even**, and

$$\Delta x = \frac{b-a}{n}$$

Example 3: Use Simpson's rule with n=4 to approximate

$$\int_0^2 5x^4 dx$$

Solution: Partition the period [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points.

x	$y = 5x^4$
0	0
½	5/16
1	5
3/2	405/16
2	80

Applying Simpson's rule with n=4 and Δx=1/2,

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

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