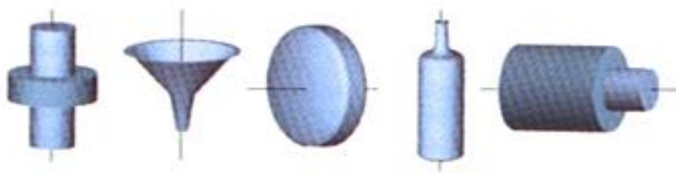
Al-Mustaqbal University / College of Engineering & Technology
Class: first
Subject: integral Mathematics/Code: UOMU024024
Lecturer: M.Sc. Alaa Khalid
Lecture name: volume revolution
Lecture: 11
2nd term

Volumes by Integration

1. **Finding volume of a solid of revolution using a disc method.**
2. **Finding volume of a solid of revolution using a washer method.**
3. **Finding volume of a solid of revolution using a shell method.**



Solids of Revolution

If a region in the plane is revolved about a given line, the resulting solid is a solid of revolution, and the line is called the axis of revolution. When calculating the volume of a solid generated by revolving a

1. Finding volume of a solid of revolution using a disc method.

The simplest solid of revolution is a right circular cylinder which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, (the disc).

To see how to calculate the volume of a general solid of revolution with a disc cross-section, using integration techniques, consider the following solid of revolution formed by revolving the plane region bounded by *f(x)*, y-axis and the vertical line x=2 about the x-axis. (see Figure1 to 4 below):

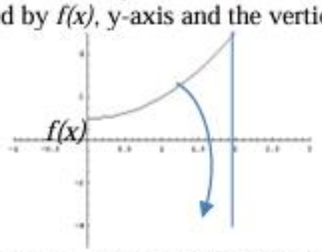


Figure 1. The area under *f(x)*, bounded by *f(x)*, x-axis, y-axis and the vertical line x=2 rotated about x-axis

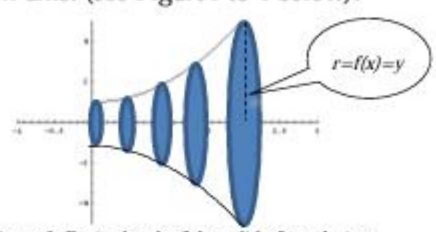


Figure 2. Basic sketch of the solid of revolution with few typical discs indicated.

Figure 3. Family of discs

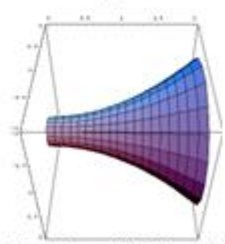


Figure 4. The 3-D model of the solid of revolution.

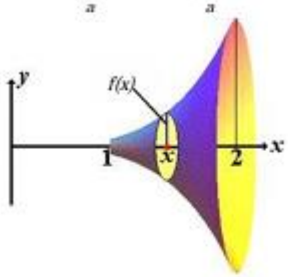
Email : alaa.khalid.abdalreda@uomus.edu.iq

FORMULAS: $V=\int Adx$, or respectively $\int Ady$ where A stands for the area of the typical disc.

Another words: $A=\pi r^2$ and $r=f(x)$ or $r=f(y)$ depending on the axis of revolution.

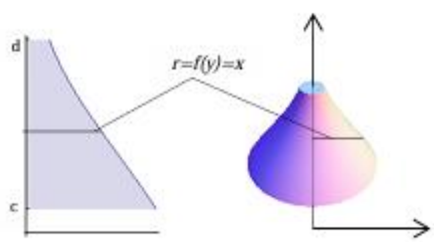
1. The volume of the solid generated by a region under $f(x)$ bounded by the x-axis and vertical lines $x=a$ and $x=b$, which is revolved **about the x-axis** is

$$V=\pi\int_a^b y^2\,dx=\pi\int_a^b\left[f(x)\right]^2 dx \qquad \text{(disc with respect to x and } r=y=f(x)\text{)}$$



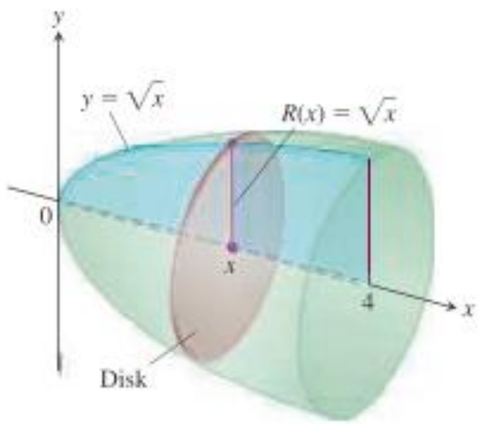
2. The volume of the solid generated by a region under $f(y)$ (to the left of $f(y)$ bounded by the y-axis, and horizontal lines $y=c$ and $y=d$ which is revolved **about the y-axis**.

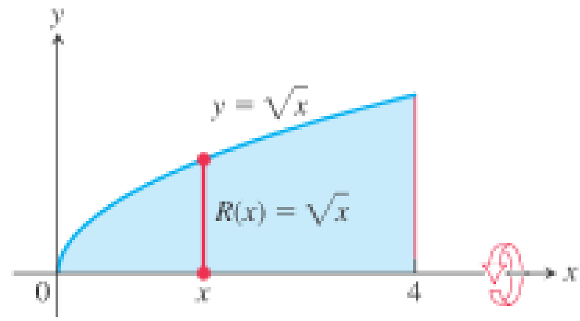
$$V=\pi\int_c^d x^2\,dy=\pi\int_c^d\left[f(y)\right]^2 dy \qquad \text{(disc with respect to y and } r=x=f(y)\text{)}$$



Email : alaa.khalid.abdalreda@uomus.edu.iq

Determine the volume of the solid generated by rotating the region





Solution

$$V = \int_a^b \pi [R(x)]^2 \, dx = \int_0^4 \pi \left[\sqrt{x}\right]^2 dx$$

$$= \pi \int_0^4 x \, dx = \pi \frac{x^2}{2} \Big]_0^4 = 8\pi$$

2. Finding volume of a solid of revolution using a washer method.

This is an extension of the disc method. The procedure is essentially the same, but now we are dealing with a hollowed object and two functions instead of one, so we have to take the difference of these functions into the account.

The general formula in this case would be:

$A = \pi\left(R^2 - r^2\right)$ where R is an outer radius and r is the inner radius.

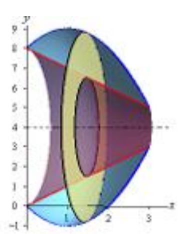
FORMULAS: V= $\int A(x)\,dx$, or respectively $\int A(y)\,dy$

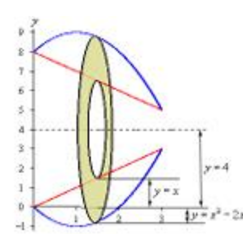
1. The volume of the solid generated by a region between *f(x)*and *g(x)* bounded by the vertical lines x=a and x=b, which is revolved about the x-axis is

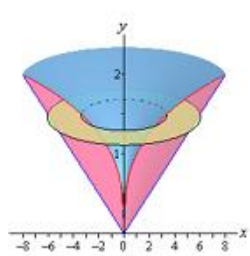
$$V = \pi\int_a^b \left|\left(f(x)\right)^2 - \left(g(x)\right)^2\right| dx \qquad \text{(washer with respect to x)}$$

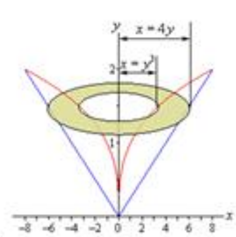
2. The volume of the solid generated by a region between *f(y)* and *g(y)* bounded by the horizontal lines y=c and y=d which is revolved about the y-axis.

$$V = \pi\int_c^d \left|\left(f(y)\right)^2 - \left(g(y)\right)^2\right| dy \qquad \text{(washer with respect to y)}$$









Email : alaa.khalid.abdalreda@uomus.edu.iq

EXAMPLE 2 Using the Washer Method

Find the volume of the solid formed by revolving the region bounded by the graphs of

$$f(x) = \sqrt{25 - x^2} \text{ and } g(x) = 3$$

about the x-axis (see Figure 5.29).

SOLUTION First find the points of intersection of f and g by setting $f(x)$ equal to $g(x)$ and solving for x.

$$f(x) = g(x) \quad \text{Set } f(x) \text{ equal to } g(x).$$

$$\sqrt{25 - x^2} = 3 \quad \text{Substitute for } f(x) \text{ and } g(x).$$

$$25 - x^2 = 9 \quad \text{Square each side.}$$

$$16 = x^2$$

$$\pm 4 = x \quad \text{Solve for } x.$$

Using $f(x)$ as the outer radius and $g(x)$ as the inner radius, you can find the volume of the solid as shown.

$$\text{Volume} = \pi \int_{-4}^{4} \{[f(x)]^2 - [g(x)]^2\}\, dx \quad \text{Washer Method}$$

$$= \pi \int_{-4}^{4} \left[\left(\sqrt{25 - x^2}\right)^2 - (3)^2\right] dx \quad \text{Substitute for } f(x) \text{ and } g(x).$$

$$= \pi \int_{-4}^{4} (16 - x^2)\, dx \quad \text{Simplify.}$$

$$= \pi \left[16x - \frac{x^3}{3}\right]_{-4}^{4} \quad \text{Find antiderivative.}$$

$$= \frac{256\pi}{3} \quad \text{Apply Fundamental Theorem.}$$

$$\approx 268.08 \quad \text{Round to two decimal places.}$$

So, the volume of the solid is about 268.08 cubic inches.

TRY IT 2

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = 1$ about the x-axis.

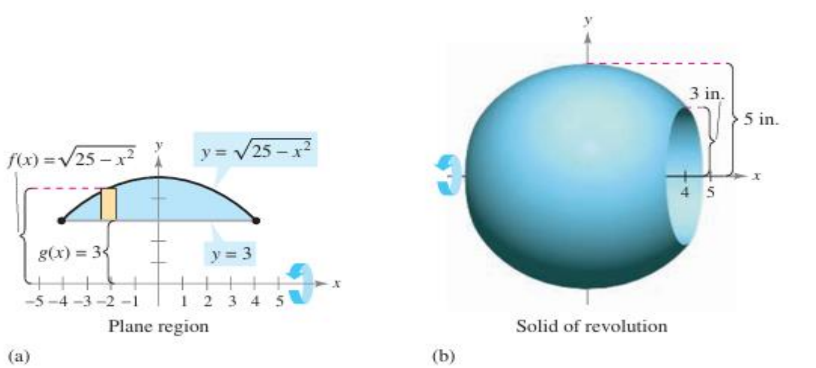


FIGURE 5.29

Email : alaa.khalid.abdalreda@uomus.edu.iq

Al-Mustaqbal University / College of Engineering & Technology
Class: first
Subject: integral Mathematics/Code: UOMU024024
Lecturer: M.Sc. Alaa Khalid
Lecture name: volume revolution
Lecture: 11
2nd term

Example: Find the volume of the solid bounded by revolving curve $y = \sqrt{x}$ &$y = x$. is revolving with

a). $x - axis$

b). $y - axis$

Solution//

a). $x - axis$ $\sqrt{x} = \mathrm{x} \rightarrow \mathrm{x} - x^2 = 0 \rightarrow x(1 - \mathrm{x}) = 0$

x=0 & x=1

$$V = \pi \int_0^1 ((\sqrt{x})^2 - x^2)dx = \pi \left(\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1) = \frac{\pi}{6} unit^3$$

b). $y - axis$ $x1 = x2 \rightarrow y = y^2 \rightarrow y - y^2 = 0$

y=0 & y=1

$$V = \pi \int_0^1 (y)^2 - y^4)\, dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15} unit^3$$

Email : alaa.khalid.abdalreda@uomus.edu.iq