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Integration

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Integration

The idea of integration is that we can compute many quantities by breaking them into small pieces, and then summing the contribution from each small part.

1. Indefinite integrals:

The set of all anti derivatives of a function is called indefinite integral of the function. Assume u and v denote differentiable functions of x , and a , n , and c are constants, then the integration formulas are:-

$$1) \int du = u(x) + c$$

$$2) \int a \cdot u(x) dx = a \int u(x) dx$$

$$3) \int (u(x) \mp v(x)) dx = \int u(x) dx \mp \int v(x) dx$$

$$4) \int u^n du = \frac{u^{n+1}}{n+1} + c \quad \text{when } n \neq -1 \quad \& \quad \int u^{-1} du = \int \frac{1}{u} du = \ln u + c$$

$$5) \int a^u du = \frac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$$

EX-1 – Evaluate the following integrals:

$$1) \int 3x^2 dx$$

$$6) \int \frac{x+3}{\sqrt{x^2+6x}} dx$$

$$2) \int \left(\frac{1}{x^2} + x \right) dx$$

$$7) \int \frac{x+2}{x^2} dx$$

$$3) \int x\sqrt{x^2+1} dx$$

$$8) \int \frac{e^x}{1+3e^x} dx$$

$$4) \int (2t + t^{-1})^2 dt$$

$$9) \int 3x^3 \cdot e^{-2x^4} dx$$

$$5) \int \sqrt{(z^2 - z^{-2})^2 + 4} dz$$

$$10) \int 2^{-4x} dx$$

Sol. –

$$1) \int 3x^2 dx = 3 \int x^2 dx = 3 \frac{x^3}{3} + c = x^3 + c$$

$$\begin{aligned}
2) \quad (x^{-2} + x)dx &= \int x^{-2} dx + \int x dx = \frac{x^{-1}}{-1} + \frac{x^2}{2} + c = -\frac{1}{x} + \frac{x^2}{2} + c \\
3) \quad \int x\sqrt{x^2 + 1} dx &= \frac{1}{2} \int 2x(x^2 + 1)^{1/2} dx = \frac{1}{2} \frac{(x^2 + 1)^{3/2}}{3/2} + c = \frac{1}{3} \sqrt{(x^2 + 1)^3} + c \\
4) \quad \int (2t + t^{-1})^2 dt &= \int (4t^2 + 4 + t^{-2}) dt = 4 \frac{t^3}{3} + 4t + \frac{t^{-1}}{-1} + c = \frac{4}{3} t^3 + 4t - \frac{1}{t} + c \\
5) \quad \int \sqrt{(z^2 - z^{-2})^2 + 4} dz &= \int \sqrt{z^4 - 2 + z^{-4} + 4} dz = \int \sqrt{z^4 + 2 + z^{-4}} dz \\
&= \int \sqrt{(z^2 + z^{-2})^2} dz = \int (z^2 + z^{-2}) dz = \frac{z^3}{3} + \frac{z^{-1}}{-1} + c = \frac{1}{3} z^3 - \frac{1}{z} + c \\
6) \quad \int \frac{x+3}{\sqrt{x^2+6x}} dx &= \frac{1}{2} \int (2x+6) \cdot (x^2+6x)^{-1/2} dx \\
&= \frac{1}{2} \cdot \frac{(x^2+6x)^{1/2}}{1/2} + c = \sqrt{x^2+6x} + c \\
7) \quad \int \frac{x+2}{x^2} dx &= \int \left(\frac{x}{x^2} + \frac{2}{x^2} \right) dx = \int (x^{-1} + 2x^{-2}) dx = \ln x + \frac{2x^{-1}}{-1} + c = \ln x - \frac{2}{x} + c \\
8) \quad \int \frac{e^x}{1+3e^x} dx &= \frac{1}{3} \int 3e^x (1+3e^x)^{-1} dx = \frac{1}{3} \ln(1+3e^x) + c \\
9) \quad \int 3x^3 \cdot e^{-2x^4} dx &= -\frac{3}{8} \int -8x^3 \cdot e^{-2x^4} dx = -\frac{3}{8} \cdot e^{-2x^4} + c \\
10) \quad \int 2^{-4x} dx &= -\frac{1}{4} \int 2^{-4x} \cdot (-4 dx) = -\frac{1}{4} \cdot 2^{-4x} \cdot \frac{1}{\ln 2} + c
\end{aligned}$$

2. Integrals of trigonometric functions:

The integration formulas for the trigonometric functions are:

$$6) \quad \int \sin u \cdot du = -\cos u + c$$

$$7) \quad \int \cos u \cdot du = \sin u + c$$

$$8) \quad \int \tan u \cdot du = -\ln|\cos u| + c$$

$$9) \quad \int \cot u \cdot du = \ln|\sin u| + c$$

$$10) \int \sec u \cdot du = \ln|\sec u + \tan u| + c$$

$$11) \int \csc u \cdot du = -\ln|\csc u + \cot u| + c$$

$$12) \int \sec^2 u \cdot du = \tan u + c$$

$$13) \int \csc^2 u \cdot du = -\cot u + c$$

$$14) \int \sec u \cdot \tan u \cdot du = \sec u + c$$

$$15) \int \csc u \cdot \cot u \cdot du = -\csc u + c$$

Example 2: Evaluate the following integrals:

$$1) \int \cos(3\theta - 1) d\theta$$

$$6) \int \frac{d\theta}{\cos^2 \theta}$$

$$2) \int x \cdot \sin(2x^2) dx$$

$$7) \int (1 - \sin^2 3t) \cdot \cos 3t dt$$

$$3) \int \cos^2(2y) \cdot \sin(2y) dy$$

$$8) \int \tan^3(5x) \cdot \sec^2(5x) dx$$

$$4) \int \sec^3 x \cdot \tan x dx$$

$$9) \int \sin^4 x \cdot \cos^3 x dx$$

$$5) \int \sqrt{2 + \sin 3t} \cdot \cos 3t dt$$

$$10) \int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} dx$$

Solution:

$$1) \frac{1}{3} \int 3 \cos(3\theta - 1) d\theta = \frac{1}{3} \sin(3\theta - 1) + c$$

$$2) \frac{1}{4} \int 4x \cdot \sin(2x^2) dx = -\frac{1}{4} \cos(2x^2) + c$$

$$3) -\frac{1}{2} \int (\cos 2y)^2 \cdot (-2 \sin 2y dy) = -\frac{1}{2} \cdot \frac{(\cos 2y)^3}{3} + c = -\frac{1}{6} (\cos 2y)^3 + c$$

$$4) \int \sec^2 x \cdot (\sec x \cdot \tan x \cdot dx) = \frac{\sec^3 x}{3} + c$$

$$5) \frac{1}{3} \int (2 + \sin 3t)^{1/2} (3 \cos 3t dt) = \frac{1}{3} \cdot \frac{(2 + \sin 3t)^{3/2}}{3/2} + c = \frac{2}{9} \sqrt{(2 + \sin 3t)^3} + c$$

$$6) \int \frac{d\theta}{\cos^2 \theta} = \int \sec^2 \theta \cdot d\theta = \tan \theta + c$$

$$7) \int (1 - \sin^2 3t) \cdot \cos 3t \, dt = \frac{1}{3} \int 3 \cos 3t \, dt - \frac{1}{3} \int (\sin 3t)^2 \cdot 3 \cos 3t \, dt$$

$$= \frac{1}{3} \sin 3t - \frac{1}{3} \cdot \frac{\sin^3 3t}{3} + c = \frac{1}{3} \cdot \sin 3t - \frac{1}{9} \sin^3 3t + c$$

$$8) \frac{1}{5} \int \tan^3 5x \cdot (5 \sec^2 5x \, dx) = \frac{1}{5} \cdot \frac{\tan^4 5x}{4} + c = \frac{1}{20} \tan^4 5x + c$$

$$9) \int \sin^4 x \cdot \cos^3 x \, dx = \int \sin^4 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx$$

$$= \int \sin^4 x \cdot \cos x \, dx - \int \sin^6 x \cdot \cos x \, dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

$$10) \int \frac{\cot^2 \sqrt{x}}{\sqrt{x}} \, dx = \int \frac{\csc^2 \sqrt{x} - 1}{\sqrt{x}} \, dx = 2 \int \frac{\csc^2 \sqrt{x}}{2\sqrt{x}} - \int x^{-1/2} \, dx$$

$$= 2 \left(-\cot \sqrt{x} \right) - \frac{x^{1/2}}{1/2} + c = -2 \cot \sqrt{x} - 2\sqrt{x} + c$$

3. Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c = -\cos^{-1} \frac{u}{a} + c \quad ; \quad \forall u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c = -\frac{1}{a} \cot^{-1} \frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{u}{a} \right| + c = -\frac{1}{a} \csc^{-1} \left| \frac{u}{a} \right| + c \quad ; \quad \forall u^2 > a^2$$

Example 3: Evaluate the following integrals:

$$1) \int \frac{x^2}{\sqrt{1-x^6}} dx$$

$$6) \int \frac{2dx}{\sqrt{x(1+x)}}$$

$$2) \int \frac{dx}{\sqrt{9-x^2}}$$

$$7) \int \frac{dx}{1+3x^2}$$

$$3) \int \frac{x}{1+x^4} dx$$

$$8) \int \frac{2\cos x}{1+\sin^2 x} dx$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx$$

$$9) \int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

$$5) \int \frac{dx}{x\sqrt{4x^2-1}}$$

$$10) \int \frac{\tan^{-1} x}{1+x^2} dx$$

Solution:

$$1) \frac{1}{3} \int \frac{1}{\sqrt{1-(x^3)^2}} (3x^2 dx) = \frac{1}{3} \sin^{-1} x^3 + c$$

$$2) \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \frac{x}{3} + c$$

$$3) \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{1}{2} \tan^{-1} x^2 + c$$

$$4) \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}} dx = \sin^{-1}(\tan x) + c$$

$$5) \int \frac{2 dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6) \int \frac{2}{\sqrt{x}(1+x)} dx = 4 \int \frac{\frac{1}{2\sqrt{x}} dx}{1+(\sqrt{x})^2} = 4 \tan^{-1} \sqrt{x} + c$$

$$7) \frac{1}{\sqrt{3}} \int \frac{\sqrt{3} dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + c$$

$$8) 2 \int \frac{\cos x dx}{1+(\sin x)^2} = 2 \tan^{-1}(\sin x) + c$$

$$9) \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10) \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

4. Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19) \int \sinh u \cdot du = \cosh u + c$$

$$20) \int \cosh u \cdot du = \sinh u + c$$

$$21) \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22) \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23) \int \sec h^2 u \cdot du = \tanh u + c$$

$$24) \int \csc h^2 u \cdot du = \coth u + c$$

$$25) \int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$$

$$26) \int \csc hu \cdot \coth u \cdot du = -\csc hu + c$$

Example 4: Evaluate the following integrals:

$$1) \int \frac{\cosh(\ln x)}{x} dx$$

$$6) \int \operatorname{sech}^2(2x-3) dx$$

$$2) \int \sinh(2x+1) dx$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$3) \int \frac{\sinh x}{\cosh^4 x} dx$$

$$8) \int (e^{ax} - e^{-ax}) dx$$

$$4) \int x \cdot \cosh(3x^2) dx$$

$$9) \int \frac{\sinh x}{1 + \cosh x} dx$$

$$5) \int \sinh^4 x \cdot \cosh x dx$$

$$10) \int \operatorname{csch}^2 x \cdot \coth x dx$$

Solution:

$$1) \int \cosh(\ln x) \cdot \left(\frac{dx}{x}\right) = \sinh(\ln x) + c$$

$$2) \frac{1}{2} \int \sinh(2x+1) \cdot (2 dx) = \frac{1}{2} \cosh(2x+1) + c$$

$$3) \int \frac{1}{\cosh^3 x} \cdot \frac{\sinh x}{\cosh x} dx = \int \operatorname{sech}^3 x \cdot \tanh x dx$$

$$= - \int \operatorname{sech}^2 x \cdot (-\operatorname{sech} x \cdot \tanh x dx) = -\frac{\operatorname{sech}^3 x}{3} + c$$

$$4) \frac{1}{6} \int \cosh(3x^2) \cdot (6x dx) = \frac{1}{6} \sinh(3x^2) + c$$

$$5) \int \sinh^4 x \cdot (\cosh x dx) = \frac{\sinh^5 x}{5} + c$$

$$6) \frac{1}{2} \int \operatorname{sech}^2(2x-3) \cdot (2 dx) = \frac{1}{2} \tanh(2x-3) + c$$

$$7) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \tanh x dx = \ln(\cosh x) + c$$

$$8) 2 \int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a} \int \sinh ax (a dx) = \frac{2}{a} \cosh ax + c$$

$$9) \int \frac{\sinh x \, dx}{1 + \cosh x} = \ln(1 + \cosh x) + c$$

$$10) - \int \csc hx \cdot (-\csc hx \cdot \coth x \, dx) = -\frac{\csc h^2 x}{2} + c$$

5. Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27) \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28) \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29) \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u| < 1 \\ \coth^{-1} u + c & \text{if } |u| > 1 \end{cases} = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| + c$$

$$30) \int \frac{du}{u\sqrt{1-u^2}} = -\sec h^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31) \int \frac{du}{u\sqrt{1+u^2}} = -\csc h^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

Example 5: Evaluate the following integrals

$$1) \int \frac{dx}{\sqrt{1+4x^2}}$$

$$2) \int \frac{dx}{\sqrt{4+x^2}}$$

$$3) \int \frac{dx}{1-x^2}$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}}$$

$$5) \int \frac{\sec^2 \theta \, d\theta}{\sqrt{\tan^2 \theta - 1}}$$

$$6) \int \tanh^{-1}(\ln \sqrt{x}) \cdot \frac{dx}{x(1 - \ln^2 \sqrt{x})}$$

Solution:

$$1) \frac{1}{2} \int \frac{2 dx}{\sqrt{1+4x^2}} = \frac{1}{2} \sinh^{-1} 2x + c$$

$$2) \int \frac{1/2 dx}{\sqrt{1+(x/2)^2}} = \sinh^{-1} \frac{x}{2} + c$$

$$3) \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x| < 1 \\ = \coth^{-1} x + c \quad \text{if } |x| > 1$$

$$4) \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2} \int \frac{1/2 dx}{x/2 \sqrt{1+(x/2)^2}} = -\frac{1}{2} \cosh^{-1} \left| \frac{x}{2} \right| + c$$

$$5) \int \frac{1}{\sqrt{\tan^2 \theta - 1}} (\sec^2 \theta d\theta) = \cosh^{-1} (\tan \theta) + c$$

$$6) \quad \text{let } u = \ln \sqrt{x} = \frac{1}{2} \ln x \quad du = \frac{1}{2x} dx$$

$$\int \tanh^{-1} (\ln \sqrt{x}) \cdot \frac{dx}{x(1-\ln^2 \sqrt{x})} = \int \tanh^{-1} u \cdot \frac{2 du}{1-u^2} \\ = 2 \frac{(\tanh^{-1} u)^2}{2} + c = [\tanh^{-1} (\ln \sqrt{x})]^2 + c$$