Integration Methods:-

(2) products and powers of Trig. Func

In this lecture, we'll go over a quite few integrals involving Trigonometric Functions & some of the techniques we can use to help evaluate them.

The typical Form of integrating products powers of sine & cosine is:

$$\int \sin^n(x) \cos^m(x)\, dx$$

We have several cases to deal with;

(1) If n is odd اذا كان n عدد فردي

In this case we can strip out one sine & convert the rest to cosines using;

$$\sin^2 x + \cos^2 x = 1$$

Then use the substitution $(u = \cos x)$

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Ex) Evaluate $\int \sin^5 x \, dx$?

Solution)

$$\int \sin^5 x \, dx = \int \sin^4 x \, \sin x \, dx$$

$$= \int (\sin^2 x)^2 \sin x \, dx \quad -(1)$$

$\because \sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x \quad -(2)$

put (2) in (1), yields;

$$\int \sin^5 x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx \quad -(3)$$

let $\boxed{u = \cos x}$ $-(4)$ $\Rightarrow$ $\boxed{du = -\sin x \, dx}$ $-(5)$

put (4) & (5) into (3), yields;

$$\therefore \int \sin^5 x \, dx = \int (1 - u^2)^2 (-du)$$

$$= -\int (1 - 2u^2 + u^4) \, du$$

$$= -\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + c$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c$$

Now back substitute on u by $\cos x$

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$$\therefore \int \sin^5 x = \boxed{-\cos x + \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$$ Ans

② IF m is odd اذا كان m عدد فردي

In this case we strip out one cosine & convert the rest to sines using $\sin^2 x + \cos^2 x = 1$

Then use the substitution $(u = \sin x)$

Ex ⑩ Evaluate $\int \cos^3 x \, dx$?

(Solution)

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx \quad \text{—} ①$$

$$\because \sin^2 x + \cos^2 x = 1 \Longrightarrow \cos^2 x = 1 - \sin^2 x \quad \text{—} ②$$

② in ①, yields;

$$\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx \quad \text{—} ③$$

let $\boxed{u = \sin x}$ —④ $\Longrightarrow \boxed{du = \cos x \, dx}$ —⑤

put ④ & ⑤ into ③, yields;

$$\int \cos^3 x \, dx = \int (1 - u^2) \, du$$

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$$= u - \frac{u^3}{3} + C$$

Now, back substitute on u by $\sin x$

$$\therefore \int \cos^3 x \, dx = \boxed{\sin x - \frac{\sin^3 x}{3} + C} \quad \text{Ans}$$

Ex) ② $\int \cos^5 x \, dx$?

Solution

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$$
$$= \int (\cos^2 x)^2 \cos x \, dx \quad \text{——} ①$$

$\because \quad \cos^2 x = 1 - \sin^2 x \quad$ —— ②

② in ① yields;

$$\int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx \quad \text{——} ③$$

let $\boxed{u = \sin x}$ —④ $\Longrightarrow$ $\boxed{du = \cos x \, dx}$ —⑤

④ & ⑤ in ③ yields;

$$\int \cos^5 x \, dx = \int (1 - u^2)^2 \, du$$
$$= \int (1 - 2u^2 + u^4) \, du$$
$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

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Back substitute on u by $\sin x$

$$\therefore \int \cos^5 x\,dx = \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}$$ Ans.

③ If n & m are odd كل n و m فردي

In this case we can strip out either sine or cosine, However, it's easier to convert the term with smaller power.

Ex) ① Evaluate $\int \sin^5 x \cos^3 x\,dx$?

solution

$$\int \sin^5 x \cos^3 x\,dx = \int \sin^5 x \cos^2 x \cos x\,dx$$

$$= \int \sin^5 x (1 - \sin^2 x)\cos x\,dx$$

$$= \int (\sin^5 x - \sin^7 x)\cos x\,dx \quad \text{——} ①$$

let $\boxed{u = \sin x}$ ② $\longrightarrow$ $\boxed{du = \cos x\,dx}$ ③

substitute ② & ③ into ①, yields;

$$\int \sin^5 x \cos^3 x\,dx = \int (u^5 - u^7)\,du$$

$$= \frac{1}{6}u^6 - \frac{1}{8}u^8 + c$$

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Back substitute on $\underline{u}$ by $\underline{\sin x}$

$$\therefore \int \sin^5 x \cos^3 x \, dx = \boxed{\frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C}$$

Ans

④ IF Both n & m Even اذا m, n عدد زوجي

In such cases, mostly, we do rewrite the integral using the following equations (identities we studied last course).

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

$$\sin x \cos x = \frac{1}{2}\sin(2x)$$

Ex) ① determine $\int \sin^2 x \, dx$?

Solution

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos(2x)) \, dx$$

$$= \frac{1}{2}\int (1 - \cos(2x)) \, dx$$

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$$\int \sin^2 x \, dx = \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C} \quad \underline{Ans}$$

Ex) ③ Evaluate $\int \cos^2(3x)\,dx$?

Solution

$$\int \cos^2(3x)\,dx = \frac{1}{2}\int (1+\cos(6x))\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(6x)}{6}\right] + C$$

$$= \boxed{\frac{1}{2}x + \frac{1}{12}\sin(6x) + C} \quad \underline{Ans}$$

Note:

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

$$\cos^2(3x) = \frac{1}{2}(1+\cos(6x))$$

Also,

$$\cos^2(2x) = \frac{1}{2}(1+\cos(4x))$$

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Ex) ③ Determine $\int \sin^4 x\, dx$?

Solution

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx = \int \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx$$

$$= \int \frac{1}{4}\left[1 - 2\cos(2x) + \cos^2(2x)\right] dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1}{2}\left[1+\cos(4x)\right]\right] dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\right] dx \quad \frac{3}{2}$$

$$= \frac{1}{4}\int \left[\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right] dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \cancel{2}\frac{\sin(2x)}{\cancel{2}} + \frac{1}{2}\frac{\sin(4x)}{4}\right] + C$$

$$= \boxed{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C} \quad \text{Ans}$$

Ex) ④ Evaluate $\int \sin^2 x \cos^2 x\, dx$

Solution

$$\int \sin^2 x \cos^2 x\, dx = \int \frac{1}{2}\left[1-\cos(2x)\right] \cdot \frac{1}{2}\left[1+\cos(2x)\right] dx$$

$$= \frac{1}{4}\int \left[1 - \cos^2(2x)\right] dx$$

$$= \frac{1}{4}\int \sin^2(2x)\, dx$$

$$= \frac{1}{4}\int \frac{1}{2}\left[1-\cos^2(4x)\right] dx$$

$$= \frac{1}{8}\int \left[1-\cos(4x)\right] dx$$

$$= \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + c$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + c} \quad \underline{Ans}$$

Ex) ⑤ $\int \sin^2 x \cos^2 x \, dx$

Solution

In Ex④ we solved this function and now will solve it by another method !

$$\because \left(\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right] \right)$$

$$\therefore \int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2}\sin(2x)\right)^2 dx$$

$$= \frac{1}{4}\int \sin^2(2x)\, dx$$

$$= \frac{1}{4}\int \frac{1}{2}\left[1-\cos(4x)\right] dx = \frac{1}{8}\int (1-\cos(4x))\, dx$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\cos(4x) + c} \quad \underline{Ans}$$

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Note)

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Ex) ⑥ Evaluate $\int \cos(15x)\cos(4x)\,dx$?

Solution

$$\int \cos(15x)\cos(4x)\,dx = \int \frac{1}{2}\left[\cos(11x)+\cos(19x)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(11x)}{11}+\frac{\sin(19x)}{19}\right]+C$$

$$= \boxed{\frac{1}{22}\sin(11x)+\frac{1}{38}\sin(19x)+C}$$ Ans

Ex) ⑦ $\int \sin^6 x\cos^3 x\,dx$

Solution

$$\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x\cos^2 x\cos x\,dx$$

$$= \int \sin^6 x\,(1-\sin^2 x)\cos x\,dx$$

$$= \int (\sin^6 x - \sin^8 x)\cos x\,dx$$

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let $\boxed{u = \sin x} \longrightarrow \boxed{du = \cos x \, dx}$

$$\therefore \int (\sin^6 x - \sin^8 x) \cos x \, dx = \int (u^6 - u^8) \, du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + c$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c} \quad \underline{\text{Ans}}$$

Ex) 8 Determine $\int \cos^2 x \tan^3 x \, dx$?

Solution

$$\int \cancel{\cos^2 x} \cdot \frac{\sin^3 x}{\cos^{\cancel{3}} x} \, dx = \int \frac{\sin^3 x}{\cos x} \, dx$$

$$= \int \frac{\sin^2 x \sin x}{\cos x} \, dx = \int \frac{(1 - \cos^2 x) \sin x}{\cos x} \, dx$$

$$= \int \left(\frac{1}{\cos x} - \frac{\cos^{\cancel{2}} x}{\cancel{\cos x}} \right) \sin x \, dx$$

$$= \int \left(\frac{1}{\cos x} - \cos x \right) \sin x \, dx \quad , \text{ let } u = \cos x$$

$$du = -\sin x \, dx$$

$$= -\int \left(\frac{1}{u} - u \right) du = -\ln u + \frac{u^2}{2} + c$$

$$= \boxed{-\ln \cos x + \frac{\cos^2 x}{2} + c} \quad \underline{\text{Ans}}$$

So far, we have covered pretty much all the possible cases involving product of sines & cosines. Now, we need to cover that involve products of tan & secant.

The typical form of integrating products of secant & tangent is :

$$\int \sec^n x \tan^m x \, dx$$

Firstly, we can convert the even power of secants to tangents and vice versa by using this equation :

$$\tan^2 x + 1 = \sec^2 x$$

Similar to how we dealt with the product power of sines & cosines fns we'll want to eventually use one of the following substitutions :

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$u = \tan x \longrightarrow du = \sec^2 x \, dx$

$u = \sec x \longrightarrow du = \sec x \tan x \, dx$

- If we use "$u = \tan x$", we'll need two secant left for the "du" to work.
- And if we use "$u = \sec x$", we'll need one secant & one tangent left over for the "du" to work.

In other wards, If n is even number then strip out two secants & convert the remaining secants to tangent.

And If m is odd number and we have at least one secant, then strip out one tangent along with one secant & convert the rest tangents to secants.

If n is even & m is odd, then we can use either cases, and it will be easier to convert the term with the smallest power.

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Examples:

① $\int \sec^9 x \tan^5 x \, dx$

Solution

since the power of tangent is odd & less than that of the power of secant, then the substitution is ($u = \sec x$), & stripping out one tan & one secant

$$\therefore \int \sec^9 x \tan^5 x \, dx = \int \sec^8 x \tan^4 x \tan x \sec x \, dx$$

$$= \int \sec^8 x (\sec^2 x - 1)^2 \tan x \sec x \, dx$$

let $u = \sec x \Longrightarrow du = \sec x \tan x \, dx$

$$\therefore = \int u^8 (u^2 - 1)^2 \, du = \int u^8 (u^4 - 2u^2 + 1) \, du$$

$$= \int (u^{12} - 2u^{10} + u^8) \, du$$

$$= \left[\frac{u^{13}}{13} - \frac{2u^{11}}{11} + \frac{u^9}{9} \right] + c$$

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Back substitute on u by $\sec x$

$$\therefore \int \sec^9 x \tan^5 x \, dx = \boxed{\frac{1}{13}\sec^{13}x - \frac{2}{11}\sec^{11}x + \frac{1}{9}\sec^9 x + C} \quad \text{Ans}$$

② $\int \sec^4 x \tan^6 x \, dx$

solution

Here the power on the tangent is even, so the substitution $u = \sec x$ will not work. The power on the secant is even so we can use the substitution $u = \tan x$ for this integral.

$$\therefore \int \sec^4 x \cdot \tan^6 x \, dx = \int \sec^2 x \tan^6 x \sec^2 x \, dx$$

$$= \int (\tan^2 x + 1) \tan^6 x \sec^2 x \, dx$$

let $u = \tan x \longrightarrow du = \sec^2 x \, dx$

$$\therefore = \int (u^2+1) u^6 \, du = \int (u^8 + u^6) \, du$$

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$$\int \sec^4 x \tan^6 x \, dx = \frac{u^9}{9} + \frac{u^7}{7} + C$$

Back substitute on u by $\tan x$

$$\therefore \int \sec^4 x \tan^6 x \, dx = \boxed{\frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C}$$ Ans

(3) $\int \tan^3 x \, dx$

Solution

The power of tangent is odd, we need to strip out one tangent & use the substitution $u = \tan x$

$$\therefore \int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

$$= \int \tan x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

let $\boxed{u = \tan x} \longrightarrow \boxed{du = \sec^2 x \, dx}$

$\therefore \boxed{dx = \dfrac{du}{\sec^2 x}}$

$$\int \tan^3 x \, dx = \int u \, du - \int \frac{\sin x}{\cos x} dx$$

$$= \frac{u^2}{2} - (-\ln \cos x) + c$$

$$= \boxed{\frac{1}{2}\tan^2 x + \ln \cos x + c}$$

or

$$= \boxed{\frac{1}{2}\tan^2 x - \ln \sec x + c}$$

Note) $\ln \cos x = -\ln \sec x$

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④ $\int \sec^3 x \, dx$

solution

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, ds$$

To solve this integration, we notice that the direct substitution is not working as the power of sec is odd, so we need to use the integration by parts. ($\int u \, dv = uv - \int v \, du$) ← Integ. By Parts

let $\boxed{u = \sec x} \longrightarrow \boxed{du = \sec x \tan x \, dx}$

& $dv = \sec^2 x dx \longrightarrow \boxed{v = \tan x}$

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$\therefore \int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$

We still have odd power secant & Even power tangent, but we can do the follows $(\tan^2 x = \sec^2 x - 1)$,

$\therefore \int \sec^3 x\, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, d \; + \int \sec x\, dx$

$2 \int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$

Here we need to solve $\int \sec x\, dx$ separately as follows:

$\int \sec x\, dx = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)}\, dx$

$= \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)}\, dx$

let $\boxed{u = \sec x + \tan x} \longrightarrow \boxed{du = (\sec x \tan x + \sec^2 x)\, dx}$

$\therefore \int \sec x\, dx = \int \frac{du}{u} = \ln u = \boxed{\ln(\sec x + \tan x)}$ Ans

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$$\therefore \int \sec^3 x = \boxed{\frac{1}{2}\left[\sec x \tan x + \ln(\sec x + \tan x)\right] + C}$$

Ans

HW

Solve :

① $\int \frac{\sin^7 x}{\cos^4 x} \, dx$

② $\int \tan^3 x \sec x \, dx$

③ $\int \sin^5 x \cos^2 x \, dx$

④ $\int \sec^2 x \tan x \, dx$

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Summary :- p 481

a- To compute $\int \sin^n x \cos^m x \, dx$

* IF $\boxed{m \text{ odd}}$ then try $\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$

* IF $\boxed{n \text{ odd}}$ then try $\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$

$\boxed{\sin^2 x + \cos^2 x = 1}$

* IF $\boxed{n \text{ \& } m \text{ Even}}$ then rewrite the integral using Trig. IDs, like:
  * $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
  * $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
  * $\sin x \cos x = \frac{1}{2}\sin(2x)$

b- To compute $\int \sec^n x \tan^m x \, dx$

* IF $\boxed{n \text{ Even}}$, stripout $\sec^2 x$ & try $\begin{cases} u = \tan x \\ du = \sec^2 x \, dx \end{cases}$

* IF $\boxed{m \text{ odd}}$, stripout $\sec x \tan x$ & try $\begin{cases} u = \sec x \\ du = \sec x \tan x \, dx \end{cases}$

$\boxed{\tan^2 x + 1 = \sec^2 x}$

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