A matrix is a rectangular array of elements (scalars) from a field. The order, or size, of a matrix is specified by the number of rows and the number of columns, i.e. A an "m by n" matrix has m rows and n columns, and the element in the i th row and j th column is often denoted by a_{ij} :

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A vector is a matrix with a single row (or column) of n elements , i.e. the column vector is:-

$$A = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} \quad \textit{and row vector is} \quad A = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix}$$

The matrix is square if the number of rows and columns are equal (i.e. $m = n$) and the elements a_{ij} of a square matrix are called the main diagonal.

The identity matrix: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is square matrix

with one in each main diagonal position and zeros else.

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The diagonal matrix $D = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n \end{bmatrix}$ has the elements $a_1, a_2, \dots, a_n$ in its main diagonal position and zeros in all other locations, some of the a_i may be zero but not all.

A $n \times n$ triangular matrix has the pattern:-

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \quad or \quad \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

lower triangular matrix *upper triangular matrix*

The $m \times n$ null matrix:- $0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$ has zero in each of its positions.

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Elementary operations with matrices and vectors

1. **Equality:-** Two $m \times n$ matrices and A and B are said to be equal if: $a_{ij} = b_{ij} \quad \forall$ *pairs of i and j.*

EX-1 – Find the values of x, y for the following matrix equation:

$$\begin{bmatrix} x-2y & 0 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & x+y \end{bmatrix}$$

Sol. –

$$\left.\begin{aligned} x-2y=3 \quad &....(1) \\ x+y=6 \quad &....(2)*2 \end{aligned}\right\} \Rightarrow \begin{array}{c} x-2y=3 \\ 2x+2y=12 \\ \hline 3x=15 \Rightarrow \boxed{x=5} \end{array}$$

$$substitution \;\; x=5 \;\; in\,(2) \Rightarrow 5+y=6 \Rightarrow \boxed{y=1}$$

2. **Addition:-** The sum of two matrices of like dimensions is the matrix of the sum of the corresponding elements. If:-

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ .... & .... & .... & .... \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix}$$

then

$$A \mp B = \begin{bmatrix} a_{11} \mp b_{11} & a_{12} \mp b_{12} & .... & a_{1n} \mp b_{1n} \\ a_{21} \mp b_{21} & a_{22} \mp b_{22} & .... & a_{2n} \mp b_{2n} \\ .... & .... & .... & .... \\ a_{m1} \mp b_{m1} & a_{m2} \mp b_{m2} & .... & a_{mn} \mp b_{mn} \end{bmatrix}$$

1) $A+B = B+A$
2) $A+(B+C) = (A+B)+C$
3) $A-(B-C) = A-B+C$

EX-2- **Find $A+B$ and $A-B$ if:-**

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

Sol.-

$$A + B = \begin{bmatrix} 2+1 & 1-2 & 3+2 \\ 1+2 & 0+3 & -2-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 3 & 3 & -3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-1 & 1-(-2) & 3-2 \\ 1-2 & 0-(+3) & -2-(-1) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

3. Multiplication by a scalar:- **The matrix A is multiplied by the scalar C by multiplying each element of A by c:-**

$$CA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

EX-3- **Assume** $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \end{bmatrix}$, **find $3A$.**

Sol.-

$$3A = \begin{bmatrix} 3*3 & 3*2 & 3*1 \\ 3*0 & 3*5 & 3*(-1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 0 & 15 & -3 \end{bmatrix}$$

4. Matrix multiplication:- For the matrix product AB to be defined it is necessary that the number of columns of A be equal to the number of rows of B. The dimensions of such matrices are said to be conformable. If A is of dimensions $m \times p$ and B is $p \times n$, then the $ij\ th$ element of the product $C = AB$ is computed as:-

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

This is the sum of the products of corresponding elements in the $i\ th$ row of A and $j\ th$ column of B. The dimensions of AB are of course $m \times n$.

EX-4- Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 5 & 4 \\ -1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ find AB.

Sol.-

$$AB = \begin{bmatrix} 1*6+2(-1)+3*0 & 1*5+2*1+3*2 & 1*4+2(-1)+3*0 \\ -1*6+0(-1)+1*0 & -1*5+0*1+1*2 & -1*4+0(-1)+1*0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 & 2 \\ -6 & -3 & -4 \end{bmatrix}$$

Properties of multiplication:-

a) $A(B+C) = AB + AC$ *distributive law*

b) $A(BC) = (AB)C$ *associative law*

c) $AB \neq BA$ *commutative law does not hold*

d) $AI = IA = A$

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EX-5- Assume $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ *and* $B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$, verify that $AB \neq BA$.

Sol.-

$$AB=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}7 & 1\\ 6 & 3\end{bmatrix} \quad \& \quad BA=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}=\begin{bmatrix}3 & 3\\ 2 & 7\end{bmatrix}$$

$$\text{Hence } AB \neq BA$$

5. **Transpose of matrix:-** Let A is any $m \times n$ matrix the transpose of A is $n \times m$ matrix A' formed by interchanging the role of rows and columns.

$$A' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

If a matrix is square and equal to its transpose, it is said to be symmetric, then $a_{ij} = a_{ji}$ for all pairs of i and j.

Properties of transpose are:-

a) $(A \mp B)' = A' \mp B'$

b) $(AB)' = B'A'$

c) $(A')' = A$

EX-6- Assume $A=\begin{bmatrix}3 & 2 & 5\\ 2 & -1 & 4\\ 5 & 4 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix}4 & -1 & 0\\ 5 & 4 & 3\\ 2 & 1 & -1\end{bmatrix}$, show that:-

1) A *is symmetric matrix* 2) $(A+B)' = A' + B'$

3) $(AB)' = B'A'$

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Sol.-

$$1)\ A' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = A \Rightarrow A \text{ is a symmetric matrix.}$$

$$2)\ L.H.S. = (A+B)' = \begin{bmatrix} 7 & 1 & 5 \\ 7 & 3 & 7 \\ 7 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix}$$

$$R.H.S. = A' + B' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix} = L.H.S.$$

$$\therefore (A+B)' = A' + B'$$

$$3)\ L.H.S. = (AB)' = \begin{bmatrix} 32 & 10 & 1 \\ 11 & -2 & -7 \\ 40 & 11 & 12 \end{bmatrix}' = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix}$$

$$R.H.S. = B'A' = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix} = L.H.S.$$

$$\therefore (AB)' = B'A'$$

Determinants

The minor of the element a_{ij} in a matrix A is the determinant of the matrix that remains when the row and column containing a_{ij} are deleted. For example, let:-

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then the minor of } a_{21} \text{ is } \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \text{then the minor of } a_{34} \text{ is } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

and so on.

The cofactor of a_{ij} is the determinant A_{ij} that is $(-1)^{i+j}$ times the minor of a_{ij} . Thus:-

$$\text{for matrix } (3\times 3) \Rightarrow A_{23}=(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{for matrix } (4\times 4) \Rightarrow A_{31}=(-1)^{3+1}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

With each square matrix A we associate a number $\det A$ or $|A|$ or $|a_{ij}|$ called the determinant of A, calculated from the entries of A in the following way:-

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$$for\ \ n=1\ ,\ A=[a] \Rightarrow |A| = a$$

$$for\ \ n=2\ ,\ A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}\,a_{22} - a_{12}\,a_{21}$$

$$for\ \ n=3\ ,\ A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

The determinant of a square matrix can be calculated from the cofactors of any row or any column.

***EX-8*- Find the determinant of the matrix:-** $A=\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

Sol.-

1st method

$$A = \left[\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 3 & -1 & -2 & 3 & -1 \\ 2 & 3 & 1 & 2 & 3 \end{array}\right.$$

$$= 2(-1)\cdot 1 + 1(-2)\cdot 2 + 3\cdot 3\cdot 3 - (3(-1)\cdot 2 + 2(-2)\cdot 3 + 1\cdot 3\cdot 1)$$
$$= 36$$

2nd method

If we were to expand the determinant by cofactors according to elements of its third column, say, we would get:-

$$A = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

$$= 3(-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 3(9-(-2)) + 2(6-2) + (-2-3) = 36$$

Useful facts about determinants:-

F-1: If two rows of matrix are identical, the determinant is zero.

EX-9 **Show that:-** $\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$

Sol.-

$$\left|\begin{matrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{matrix}\right|\begin{matrix} 3 & -1 \\ 2 & -3 \\ 3 & -1 \end{matrix} \quad = -18-15-4-(-18-15-4) = 0$$

F-2: Interchanging two rows of matrix changes the sign of its determinants.

EX-10 **Show that:-** $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \left|\begin{matrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{matrix}\right|\begin{matrix} 2 & 1 \\ -1 & 0 \\ 1 & -2 \end{matrix} \quad = 0+3+10-(0-12-4) = 29$$

$$R.H.S. = \left|\begin{matrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{matrix}\right|\begin{matrix} -1 & 0 \\ 2 & 1 \\ 1 & -2 \end{matrix} \quad = -\left(-4+0-12-(3+10+0)\right) = 29 = L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$$

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F-3: The determinant of the transpose of a matrix is equal to the original determinant.

EX-11 Show that:- $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = 29 \quad from\ ex-10$$

$$R.H.S. = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}\begin{matrix} 2 & -1 \\ 1 & 0 \\ 5 & 3 \end{matrix} = 0 + 10 + 3 - (0 - 12 - 4) = 29 = L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$$

F-4: If each element of same row (or column) of a matrix is multiplied by a constant C, the determinant is multiplied by C.

EX-12 Show that:- $\begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}\begin{matrix} 6 & 3 \\ -1 & 0 \\ 1 & -2 \end{matrix} = 0 + 9 + 30 - (0 - 36 - 12) = 87$$

$$R.H.S. = 3 * 29 = 87 = L.H.S.$$

$$\therefore \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

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F-5: If all elements of a matrix above the main diagonal (or all below it) are zero, the determinant of the matrix is the product of the elements on the main diagonal.

EX-13 **Find:-** $\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}$

Sol.-

$$\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}\begin{matrix} 5 & 0 \\ 2 & 3 \\ 1 & -1 \end{matrix} = 60+0+0-(0-0-0)=60$$

$$Or\ directly \quad 5*3*4=60$$

F-6: If each element of a row of a matrix is multiplied by a constant C and the results added to a different row, the determinant is not changed.

EX-14 **Show that** $|A|=|B|$ *if* $A=\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{bmatrix}$ **and** B **is the matrix resultant from multiplying row (1) by** 2 **and adding to row (3).**

i.e. $B=\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{bmatrix}$

Sol.-

$$|A|=29 \quad from\ ex-10$$

$$|B|=\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & 0 \\ 5 & 0 \end{matrix} = 0+15+0-(0-0-14)=29$$

$$\therefore |A|=|B|$$

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EX-15 **Find** $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$

Sol.-

$$\overset{-2R_1+R_2}{\Rightarrow\Rightarrow\Rightarrow} \begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

Expanding the determinant by using the first column. $\Rightarrow\Rightarrow\Rightarrow 1\cdot\begin{vmatrix} 5 & 2 & 0 \\ 0 & 4 & -1 \\ 1 & -6 & 1 \end{vmatrix}\begin{matrix} 5 & 2 \\ 0 & 4 \\ 1 & -6 \end{matrix} \quad = 20+6+0-(0-10+0)=36$

Linear Equations

There are many methods to solve a system of linear equations:

$AX=B$

I) Row Reduction method **It is often possible to transform the linear equations step by step into an equivalent system of equations that is so simple it can be solved by inspection.**

We start with $n\times(n+1)$ **matrix** $[A \vdots B]$ **whose first** n **columns are the columns of** A **and whose last column is** B**. We are going to transform this augmented matrix with a sequence of elementary row operations into** $[I \vdots S]$ **where** S **is the solution of** X**.**

<u>*EX-16*</u> **Solve the following linear equation:**

$$\begin{aligned} 2x+3y-4z&=-3 \\ x+2y+3z&=3 \\ 3x-y-z&=6 \end{aligned}$$

<u>*Sol.*</u>

$$AX=B \quad \text{where} \quad A=\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}, \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B=\begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$$

$$[A \vdots B]=\left[\begin{array}{ccc|c} 2 & 3 & -4 & -3 \\ 1 & 2 & 3 & 3 \\ 3 & -1 & -1 & 6 \end{array}\right] \xrightarrow[-3R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} 0 & -1 & -10 & -9 \\ 1 & 2 & 3 & 3 \\ 0 & -7 & -10 & -3 \end{array}\right]$$

$$\xrightarrow[R_1 \text{ and } R_2]{\text{inter change}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & -7 & -10 & -3 \end{array}\right] \xrightarrow[-7R_2+R_3]{2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -17 & -15 \\ 0 & -1 & -10 & -9 \\ 0 & 0 & 60 & 60 \end{array}\right]$$

$$\xrightarrow[\frac{1}{6}R_3+R_2]{\frac{17}{60}R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 60 & 60 \end{array}\right] \xrightarrow[R_3*\frac{1}{60}]{R_2*(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] = [I \vdots S]$$

$$\text{Hence} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x=2 \;,\; y=-1 \;,\; z=1}$$

II) <u>Cramer's Rule</u> When the determinant of the coefficient matrix A of the system $AX=B$ is not zero (i.e. $|A| \neq 0$) the system has a unique solution that it may be found from the formulas:

$$X_i = \frac{|A_i|}{|A|}$$

Where $|A_i|$ is the determinant of the matrix, comes from replacing the $i\,th$ column in A by the column of constant B.

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EX-17 **Resolve example 16 using Cramer's rule:**

Sol.

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = -4+27+4-(-24-6-3) = 60$$

$$|A_1| = \begin{vmatrix} -3 & 3 & -4 \\ 3 & 2 & 3 \\ 6 & -1 & -1 \end{vmatrix}\begin{matrix} -3 & 3 \\ 3 & 2 \\ 6 & -1 \end{matrix} = 6+54+12-(-48+9-9) = 120$$

$$|A_2| = \begin{vmatrix} 2 & -3 & -4 \\ 1 & 3 & 3 \\ 3 & 6 & -1 \end{vmatrix}\begin{matrix} 2 & -3 \\ 1 & 3 \\ 3 & 6 \end{matrix} = -6-27-24-(-36+36+3) = -60$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 2 & 3 \\ 3 & -1 & 6 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = 24+27+3-(-18-6+18) = 60$$

$$\therefore \quad x = \frac{|A_1|}{|A|} = \frac{120}{60} \Rightarrow \boxed{x = 2}$$

$$y = \frac{|A_2|}{|A|} = \frac{-60}{60} \Rightarrow \boxed{y = -1}$$

$$z = \frac{|A_3|}{|A|} = \frac{60}{60} \Rightarrow \boxed{z = 1}$$

Problems

1) Let $A=\begin{bmatrix}1 & 2\\ 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3\\ -1 & 4 & -2\end{bmatrix}$, $C=\begin{bmatrix}3 & 1\\ 4 & -1\\ 0 & 2\end{bmatrix}$,

$D=\begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & -1 & 1\end{bmatrix}$, $E=\begin{bmatrix}3 & -1\\ 4 & 2\end{bmatrix}$. Find:-

a) AB *b)* DC *c)* $(D+I)C$ *d)* $DC+C$ *e)* DCB
f) EI *g)* $3A+E$ *h)* $-5E+A$ *i)* $E(2B)$

$$\left(ans.: a)\begin{bmatrix}-1 & 10 & -1\\ -4 & 16 & -8\end{bmatrix} b)\begin{bmatrix}3 & 9\\ 4 & 3\\ -4 & 3\end{bmatrix} c,d)\begin{bmatrix}6 & 10\\ 8 & 2\\ -4 & 5\end{bmatrix} e)\begin{bmatrix}-6 & 42 & -9\\ 1 & 20 & 6\\ -7 & 4 & -18\end{bmatrix}\right.$$
$$\left. f)\begin{bmatrix}3 & -1\\ 4 & 2\end{bmatrix} g)\begin{bmatrix}6 & 5\\ 4 & 14\end{bmatrix} h)\begin{bmatrix}-14 & 7\\ -20 & -6\end{bmatrix} i)\begin{bmatrix}8 & 4 & 22\\ 4 & 32 & 16\end{bmatrix}\right)$$

2) Find the value of x :-

$$\begin{bmatrix}x & 4 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\ -7\\ 5/4\end{bmatrix}=0$$

$$\left(ans.:\ x=\frac{1}{2}\ \ or\ \ x=1\right)$$

3) Find v and w if: $\begin{bmatrix}5 & w\end{bmatrix}=v\begin{bmatrix}-2 & 1\end{bmatrix}$.

$$\left(ans.:\ w=v=-\frac{5}{2}\right)$$

4) Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}0 & 2\\ -1 & 3\\ 5 & -2\end{bmatrix}$, Find:-

a) $2A+B'$ *b)* $B'A'-I$

$$\left(ans.: a)\begin{bmatrix}2 & -3 & 9\\ 2 & 5 & 6\end{bmatrix}\ b)\begin{bmatrix}10 & 19\\ -5 & -6\end{bmatrix}\right)$$

5) Let $A=\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & 2\\ -1 & 1 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & 0\\ -1 & 3 & 2\end{bmatrix}$, **Find:-**

$(2A-I)B'$ **and show that** $(AB)'=B'A'$

$$\left(ans.: \begin{bmatrix}5 & -1\\ -2 & 11\\ -6 & 26\end{bmatrix}\right)$$

6) **For what value of** x **will:** $\begin{vmatrix}x & x & 1\\ 2 & 0 & 5\\ 6 & 7 & 1\end{vmatrix}=0$?

$$(ans.:\ x=2)$$

7) **Let** A **be an arbitrary** 3 **by** 3 **matrix and let** R_{12} **be the matrix obtained from the** 3 **by** 3 **identity matrix by interchanging row 1 and 2 :** $R_{12}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$. **a) Compute** $R_{12}A$ **and show that you would get the same result by interchanging rows 1 and 2 of** A**. b) Compute** AR_{12} **and show that the result is that you would get by interchanging column 1 and 2 of** A**.**

$$\left(ans.:\ a)\begin{bmatrix}a_{21} & a_{22} & a_{23}\\ a_{11} & a_{12} & a_{13}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\ b)\begin{bmatrix}a_{12} & a_{11} & a_{13}\\ a_{22} & a_{21} & a_{23}\\ a_{32} & a_{31} & a_{33}\end{bmatrix}\right)$$

8) Solve the following determinants:-

$$a)\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix} \qquad b)\begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix} \qquad c)\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$d)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} \qquad e)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \qquad f)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$g)\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & - & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix} \qquad h)\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$\left(\begin{array}{llll} ans.:\ a)-5 & b)0 & c)-7 & d)6 \\ \quad e)38 & f)1 & g)2 & h)-1 \end{array} \right)$$

9) Solve the following system of equations:-

$$a)\ \begin{aligned} x + 8y &= 4 \\ 3x - y &= -13 \end{aligned} \qquad b)\ \begin{aligned} 2x + 3y &= 5 \\ 3x - y &= 2 \end{aligned} \qquad c)\ \begin{aligned} x + y + z &= 2 \\ 2x - y + z &= 0 \\ x + 2y - z &= 4 \end{aligned}$$

$$d)\ \begin{aligned} 2x + y - z &= 2 \\ x - y + z &= 7 \\ 2x + 2y + z &= 4 \end{aligned} \qquad e)\ \begin{aligned} 2x - 4y &= 6 \\ x + y + z &= 1 \\ 5y + 7z &= 10 \end{aligned} \qquad f)\ \begin{aligned} x - z &= 3 \\ 2y - 2z &= 2 \\ 2x + z &= 3 \end{aligned}$$

$$g)\ \begin{aligned} x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 - x_2 + x_3 + x_4 &= -1 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_3 + x_4 &= -1 \end{aligned} \qquad h)\ \begin{aligned} 2x - 3y + 4z &= -19 \\ 6x + 4y - 2z &= 8 \\ x + 5y + 4z &= 23 \end{aligned}$$

$$\left(\begin{array}{ll} ans.:\ a)\ x=-4\ ,\ y=1 & b)\ x=y=1 \\ c)\ x=\frac{6}{7}\ ,\ y=\frac{10}{7}\ ,\ z=-\frac{2}{7} & d)\ x=3\ ,\ y=-2\ ,\ z=2 \\ e)\ x=0\ ,\ y=-\frac{3}{2}\ ,\ z=\frac{5}{2} & f)\ x=2\ ,\ y=0\ ,\ z=-1 \\ g)\ x_1=2\ ,\ x_2=0\ ,\ x_3=x_4=-\frac{3}{2} & h)\ x=-2\ ,\ y=5\ ,\ z=0 \end{array} \right)$$

Reference

1-Nacy and Haabeeb, "Lectures Mathematics for 1st Class student, Technology University.