First Order Differential Equations

By

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Terminology and Separable Equations

A differential equation is *separable* if it can be written (perhaps after some algebraic manipulation) as

$$\frac{dy}{dx} = F(x)G(y)$$

in which the derivative equals a product of a function just of x and a function just of y. This suggests a method of solution.

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Step 1. For y such that $G(y) \neq 0$, write the differential form

$$\frac{1}{G(y)}\,dy = F(x)\,dx.$$

In this equation, we say that the variables have been separated.

Step 2. Integrate

$$\int \frac{1}{G(y)}\,dy = \int F(x)\,dx.$$

Step 3. Attempt to solve the resulting equation for y in terms of x. If this is possible, we have an explicit solution (as in Examples 1.1 through 1.3). If this is not possible, the solution is implicitly defined by an equation involving x and y (as in Example 1.4).

Step 4. Following this, go back and check the differential equation for any values of y such that $G(y) = 0$. Such values of y were excluded in writing $1/G(y)$ in step (1) and may lead to additional solutions beyond those found in step (3). This happens in Example 1.1.

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EXAMPLE 1.1

To solve $y' = y^2 e^{-x}$, first write

$$\frac{dy}{dx} = y^2 e^{-x}.$$

If $y \neq 0$, this has the differential form

$$\frac{1}{y^2} dy = e^{-x} dx.$$

The variables have been separated. Integrate

$$\int \frac{1}{y^2} dy = \int e^{-x} dx$$

or

$$-\frac{1}{y} = -e^{-x} + k$$

in which k is a constant of integration. Solve for y to get

$$y(x) = \frac{1}{e^{-x} - k}.$$

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EXAMPLE 1.2

$x^2 y' = 1 + y$ is separable, since we can write

$$\frac{1}{1+y}\,dy = \frac{1}{x^2}\,dx$$

if $y \neq -1$ and $x \neq 0$. Integrate to obtain

$$\ln|1+y| = -\frac{1}{x} + k$$

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EXAMPLE 1.4

We will solve the initial value problem

$$y' = y\frac{(x-1)^2}{y+3}; \qquad y(3) = -1.$$

The differential equation itself (not the algebra of separating the variables) requires that $y \neq -3$.

In differential form,

$$\frac{y+3}{y}\,dy = (x-1)^2\,dx$$

or

$$\left(1+\frac{3}{y}\right)dy = (x-1)^2 dx.$$

Integrate to obtain

$$y + 3\ln|y| = \frac{1}{3}(x-1)^3 + k.$$

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This does not prevent us from solving the initial value problem. We need $y(3) = -1$, so put $x = 3$ and $y = -1$ into the implicitly defined general solution to get

$$-1 = \frac{1}{3}\left(2^3\right) + k.$$

Then $k = -11/3$, and the solution of the initial value problem is implicitly defined by

$$y + 3\ln|y| = \frac{1}{3}(x-1)^3 - \frac{11}{3}.$$

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In each of Problems 7 through 16, determine if the differential equation is separable. If it is, find the general solution (perhaps implicitly defined) and also any singular solutions the equation might have. If it is not separable, do not attempt a solution.

7. $3y' = 4x/y^2$
8. $y + xy' = 0$
9. $\cos(y)\,y' = \sin(x + y)$
10. $e^{x+y}y' = 3x$
11. $xy' + y = y^2$
12. $y' = \dfrac{(x+1)^2 - 2y}{2^y}$
13. $x\sin(y)\,y' = \cos(y)$
14. $\dfrac{x}{y}y' = \dfrac{2y^2 + 1}{x + 1}$
15. $y + y' = e^x - \sin(y)$
16. $[\cos(x + y) + \sin(x - y)]y' = \cos(2x)$

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Linear Equations

A first-order differential equation is *linear* if it has the form

$$y' + p(x)y = q(x)$$

for some functions p and q.

There is a general approach to solving a linear equation. Let

$$g(x) = e^{\int p(x)\,dx}$$

and notice that

$$g'(x) = p(x)e^{\int p(x)\,dx} = p(x)g(x). \tag{1.3}$$

Now multiply $y' + p(x)y = q(x)$ by $g(x)$ to obtain

$$g(x)y' + p(x)g(x)y = q(x)g(x).$$

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In view of equation (1.3), this is

$$g(x)y' + g'(x)y = q(x)g(x).$$

Now we see the point to multiplying the differential equation by $g(x)$. The left side of the new equation is the derivative of $g(x)y$. The differential equation has become

$$\frac{d}{dx}(g(x)y) = q(x)g(x),$$

which we can integrate to obtain

$$g(x)y = \int q(x)g(x)dx + c.$$

If $g(x) \neq 0$, we can solve this equation for y:

$$y(x) = \frac{1}{g(x)}\int q(x)g(x)dx + \frac{c}{g(x)}.$$

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This is the general solution with the arbitrary constant c.

We do not recommend memorizing this formula for $y(x)$. Instead, carry out the following procedure.

Step 1. If the differential equation is linear, $y' + p(x)y = q(x)$. First compute

$$e^{\int p(x)\,dx}.$$

This is called an *integrating factor* for the linear equation.

Step 2. Multiply the differential equation by the integrating factor.

Step 3. Write the left side of the resulting equation as the derivative of the product of y and the integrating factor. The integrating factor is designed to make this possible. The right side is a function of just x.

Step 4. Integrate both sides of this equation and solve the resulting equation for y, obtaining the general solution. The resulting general solution may involve integrals (such as $\int \cos(x^2)\,dx$) which cannot be evaluated in elementary form.

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EXAMPLE 1.8

The equation $y' + y = x$ is linear with $p(x) = 1$ and $q(x) = x$. An integrating factor is

$$e^{\int p(x)dx} = e^{\int dx} = e^x.$$

Multiply the differential equation by e^x to get

$$e^x y' + e^x y = xe^x.$$

This is

$$(ye^x)' = xe^x$$

with the left side as a derivative. Integrate this equation to obtain

$$ye^x = \int xe^x dx = xe^x - e^x + c.$$

Finally, solve for y by multiplying this equation by e^{-x}:

$$y = x - 1 + ce^{-x}.$$

This is the general solution, containing one arbitrary constant. ♦

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EXAMPLE 1.9

Solve the initial value problem

$$y' = 3x^2 - \frac{y}{x};\ y(1) = 5.$$

This differential equation is not linear. Write it as

$$y' + \frac{1}{x}y = 3x^2,$$

which is linear. An integrating factor is

$$e^{\int (1/x)dx} = e^{\ln(x)} = x$$

for $x > 0$. Multiply the differential equation by x to obtain

$$xy' + y = 3x^3$$

or

$$(xy)' = 3x^3.$$

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Integrate to obtain

$$xy = \frac{3}{4}x^4 + c.$$

Solve for y to write the general solution

$$y = \frac{3}{4}x^3 + \frac{c}{x}$$

for $x > 0$. For the initial condition, we need

$$y(1) = \frac{3}{4} + c = 5.$$

Then $c = 17/4$, and the solution of the initial value problem is

$$y = \frac{3}{4}x^3 + \frac{17}{4x}.$$

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PROBLEMS

In each of Problems 1 through 5, find the general solution.

1. $y' - \frac{3}{x}y = 2x^2$
2. $y' + y = \frac{1}{2}(e^x - e^{-x})$
3. $y' + 2y = x$
4. $y' + \sec(x)y = \cos(x)$
5. $y' - 2y = -8x^2$

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In each of Problems 6 through 10, solve the initial value problem.

6. $y' + 3y = 5e^{2x} - 6;\ y(0) = 2$
7. $y' + \frac{1}{x-2}y = 3x;\ y(3) = 4$
8. $y' - y = 2e^{4x};\ y(0) = -3$
9. $y' + \frac{2}{x+1}y = 3;\ y(0) = 5$
10. $y' + \frac{5y}{9x} = 3x^3 + x;\ y(-1) = 4$

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Exact Equations

A differential equation $M(x, y) + N(x, y)y' = 0$ can be written in differential form as

$$M(x, y)dx + N(x, y)dy = 0. \tag{1.4}$$

Sometimes this differential form is the key to writing a general solution. Recall that the differential of a function $\varphi(x, y)$ of two variables is

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy. \tag{1.5}$$

If we can find a function $\varphi(x, y)$ such that

$$\frac{\partial \varphi}{\partial x} = M(x, y) \text{ and } \frac{\partial \varphi}{\partial y} = N(x, y), \tag{1.6}$$

then the differential equation $Mdx + Ndy = 0$ is just

$$M(x, y)dx + N(x, y)dy = d\varphi = 0.$$

But if $d\varphi = 0$, then $\varphi(x, y) =$ constant. The equation

$$\varphi(x, y) = c,$$

with c an arbitrary constant, implicitly defines the general solution of $Mdx + Ndy = 0$.

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EXAMPLE 1.12

Consider

$$\frac{dy}{dx} = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}.$$

This is neither linear nor separable. Write

$$(2xy^3 + 2)\,dx + (3x^2y^2 + 8e^{4y})\,dy = 0$$

so

$$M(x, y) = 2xy^3 + 2 \text{ and } N(x, y) = 3x^2y^2 + 8e^{4y}.$$

From equations (1.6), we want $\varphi(x, y)$ such that

$$\frac{\partial \varphi}{\partial x} = 2xy^3 + 2 \text{ and } \frac{\partial \varphi}{\partial y} = 3x^2y^2 + 8e^{4y}.$$

Choose either of these equations and integrate it. If we choose the first equation, then integrate with respect to x:

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$$\varphi(x, y) = \int \frac{\partial \varphi}{\partial x} dx$$

$$= \int (2xy^3 + 2)\, dx = x^2 y^3 + 2x + g(y).$$

In this integration, we are reversing a partial derivative with respect to x, so y is treated like a constant. This means that the constant of integration may also involve y; hence it is called $g(y)$. Now we know $\varphi(x, y)$ to within this unknown function $g(y)$. To determine $g(y)$, use the fact that we know what $\partial\varphi/\partial y$ must be

$$\frac{\partial \varphi}{\partial y} = 3x^2 y^2 + 8e^{4y}$$

$$= 3x^2 y^2 + g'(y).$$

This means that $g'(y) = 8e^{4y}$, so $g(y) = 2e^{4y}$. This fills in the missing piece, and

$$\varphi(x, y) = x^2 y^3 + 2x + 2e^{4y}.$$

The general solution of the differential equation is implicitly defined by

$$x^2 y^3 + 2x + 2e^{4y} = c,$$

PROBLEMS

In each of Problems 1 through 5, test the differential equation for exactness. If it is exact (on some region of the plane), find a potential function and the general solution (perhaps implicitly defined). If it is not exact anywhere, do not attempt a solution.

1. $2y^2 + ye^{xy} + (4xy + xe^{xy} + 2y)y' = 0$
2. $4xy + 2x + (2x^2 + 3y^2)y' = 0$
3. $4xy + 2x^2y + (2x^2 + 3y^2)y' = 0$
4. $2\cos(x+y) - 2x\sin(x+y) - 2x\sin(x+y)y' = 0$
5. $1/x + y + (3y^2 + x)y' = 0$

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The Homogeneous Differential Equation

A *homogeneous* differential equation is one of the form

$$y' = f(y/x)$$

with y' isolated on one side and on the other an expression in which x and y always occur in the combination y/x. Examples are $y' = \sin(y/x) - x/y$ and $y' = x^2/y^2$.

In some instances, a differential equation can be manipulated into homogeneous form. For example, with

$$y' = \frac{y}{x+y}$$

we can divide numerator and denominator on the right by x to obtain the homogeneous equation

$$y' = \frac{y/x}{1+y/x}.$$

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This manipulation requires the assumption that $x \neq 0$.

A homogeneous differential equation can always be transformed to a separable equation by letting

$$y = ux.$$

To see this, compute $y' = u'x + u$ and write $u = y/x$ to transform

$$y' = u'x + u = f(y/x) = f(u).$$

In terms of u and x, this is

$$xu' + u = f(u)$$

or

$$x\frac{du}{dx} = f(u) - u.$$

The variables u and x separate as

$$\frac{1}{f(u) - u}\, du = \frac{1}{x}\, dx.$$

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EXAMPLE 1.15

We will solve

$$xy' = \frac{y^2}{x} + y.$$

Write this as

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

With $y = ux$, this becomes

$$xu' + u = u^2 + u$$

or

$$xu' = u^2.$$

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The variables separate as

$$\frac{1}{u^2}\,du = \frac{1}{x}\,dx.$$

Integrate to obtain

$$-\frac{1}{u} = \ln|x| + c.$$

Then

$$u = \frac{-1}{\ln|x| + c}.$$

Then

$$y = \frac{-x}{\ln|x| + c},$$

and this is the general solution of the original homogeneous equation. ♦

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PROBLEMS

1. $y' = y/(2x + y)$.
2. $y' = (2xy + y^2)/(3x^2)$.
3. $y' = (2x^2 + y^2)/xy$.
4. $y' = (2xy + y^2)/x^2$.
5. $y' = (x - y)/(x + 2y)$.

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