Integration by Parts

The formula for integration by parts comes from the product rule:-

$$d(u\cdot v) = u\cdot dv + v\cdot du \quad \Rightarrow \quad u\cdot dv = d(u\cdot v) - v\cdot du$$

and integrated to give: $\int u\,dv = \int d(u\cdot v) - \int v\,du$

then the integration by parts formula is:-

$$\int u\,dv = u\cdot v - \int v\,du$$

Rule for choosing u and dv is:

For u: choose something that becomes simpler when differentiated.

For dv: choose something whose integral is simple.

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Email: firas.thair.almaliky@uomus.edu.iq

EXAMPLE 2 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

EXAMPLE 3 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$ ■

Integration by Parts

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize

the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 4 Using Tabular Integration

Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Email: firas.thair.almaliky@uomus.edu.iq

EXAMPLE 5 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Evaluate the following integrals:

1) $\int \frac{x^3}{x-1} dx$ (ans.: $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \ln(x-1) + c$)

2) $\int \frac{3x+2}{3x-1} dx$ (ans.: $x + \ln(3x - 1) + c$)

3) $\int x^2 \cdot e^{-x} dx$ (ans.: $-e^{-x}(x^2 + 2x + 2) + c$)

4) $\int x \cdot \sin x^2 dx$ (ans.: $-\frac{1}{2}\cos x^2 + c$)

Email: firas.thair.almaliky@uomus.edu.iq