Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

Subject Name: ---------------

2nd Class, Second Semester

Subject Code: [Insert Subject Code Here]

Academic Year: 2024-2025

Lecturer: Dr. or Assist lect. –Zahraa Emad---------

Email: -----------@uomus.edu.iq

Lecture No.:- 1

Lecture Title: [Insert Lecture Title Here]

Lecture QR

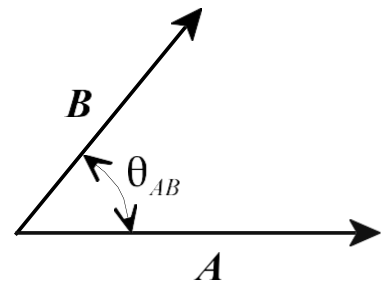
Dot Product

(Scalar Product)

The *dot product* of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ (denoted by $\boldsymbol{A} \cdot \boldsymbol{B}$) is defined as the product of the vector magnitudes and the cosine of the smaller angle between them.

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos\theta_{AB} = AB\cos\theta_{AB}$$

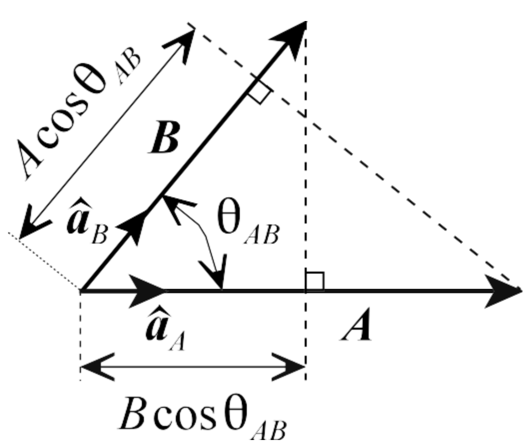
$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A} \qquad \text{(commutative law)}$$



The dot product is commonly used to determine the component of a vector in a particular direction. The dot product of a vector with a unit vector yields the component of the vector in the direction of the unit vector. Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ with corresponding unit vectors $\boldsymbol{a}_A$ and $\boldsymbol{a}_B$, the component of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$ (the *projection* of $\boldsymbol{A}$ onto $\boldsymbol{B}$) is found evaluating the dot product of $\boldsymbol{A}$ with $\boldsymbol{a}_B$. Similarly, the component of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$ (the *projection* of $\boldsymbol{B}$ onto $\boldsymbol{A}$) is found evaluating the dot product of $\boldsymbol{B}$ with $\boldsymbol{a}_A$.

$$\boldsymbol{A} \cdot \hat{\boldsymbol{a}}_B = |\boldsymbol{A}|\,|\hat{\boldsymbol{a}}_B| \cos\theta_{AB} = A\cos\theta_{AB}$$

$$\boldsymbol{B} \cdot \hat{\boldsymbol{a}}_A = |\boldsymbol{B}|\,|\hat{\boldsymbol{a}}_A| \cos\theta_{AB} = B\cos\theta_{AB}$$



The dot product can be expressed independent of angles through the use of component vectors in an orthogonal coordinate system.

$$\boldsymbol{A} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = B_x \hat{\boldsymbol{x}} + B_y \hat{\boldsymbol{y}} + B_z \hat{\boldsymbol{z}}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = (A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}}) \cdot (B_x \hat{\boldsymbol{x}} + B_y \hat{\boldsymbol{y}} + B_z \hat{\boldsymbol{z}})$$

$$= A_x B_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} + A_x B_y \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} + A_x B_z \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{z}}$$

$$+ A_y B_x \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{x}} + A_y B_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} + A_y B_z \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}}$$

$$+ A_z B_x \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}} + A_z B_y \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{y}} + A_z B_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}}$$

The dot product of like unit vectors yields one ($\theta_{AB} = 0^o$) while the dot product of unlike unit vectors ($\theta_{AB} = 90^o$) yields zero. The dot product results are

$$\begin{array}{lll} \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}} = 1 & \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = 0 & \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{z}} = 0 \\ \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{x}} = 0 & \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{y}} = 1 & \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} = 0 \\ \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{x}} = 0 & \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{y}} = 0 & \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = 1 \end{array}$$

The resulting dot product expression is

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$$

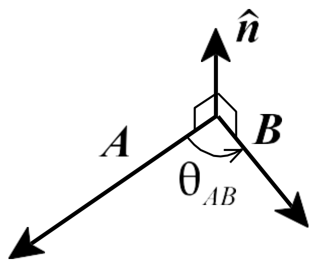
Cross Product

(Vector Product)

The *cross product* of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ (denoted by $\boldsymbol{A}\times\boldsymbol{B}$) is defined as the product of the vector magnitudes and the sine of the smaller angle between them with a vector direction defined by the *right hand rule*.

$$\boldsymbol{A}\times\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta_{AB}\hat{\boldsymbol{n}} = AB\sin\theta_{AB}\hat{\boldsymbol{n}}$$

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A} \qquad \text{(not commutative)}$$



Note: (1) the unit vector $\hat{\boldsymbol{n}}$ is normal to the plane in which $\boldsymbol{A}$ and $\boldsymbol{B}$ lie.

(2) $AB\sin\theta_{AB}$ = area of the parallelogram formed by the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

Using component vectors, the cross product of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be written as

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}$$

$$\boldsymbol{A}\times\boldsymbol{B} = (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}})\times(B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}})$$

$$= A_xB_x\hat{\boldsymbol{x}}\times\hat{\boldsymbol{x}} + A_xB_y\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} + A_xB_z\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}}$$

$$+ A_yB_x\hat{\boldsymbol{y}}\times\hat{\boldsymbol{x}} + A_yB_y\hat{\boldsymbol{y}}\times\hat{\boldsymbol{y}} + A_yB_z\hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}}$$

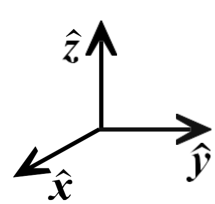
$$+ A_zB_x\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}} + A_zB_y\hat{\boldsymbol{z}}\times\hat{\boldsymbol{y}} + A_zB_z\hat{\boldsymbol{z}}\times\hat{\boldsymbol{z}}$$

The cross product of like unit vectors yields zero ($\theta_{AB} = 0^{o}$) while the cross product of unlike unit vectors ($\theta_{AB} = 90^{o}$) yields another unit vector which is determined according to the right hand rule. The cross products results are

$$\hat{\boldsymbol{x}}\times\hat{\boldsymbol{x}} = 0 \qquad \hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}} = -\hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{y}}\times\hat{\boldsymbol{x}} = -\hat{\boldsymbol{z}} \qquad \hat{\boldsymbol{y}}\times\hat{\boldsymbol{y}} = 0 \qquad \hat{\boldsymbol{y}}\times\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}$$

$$\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \qquad \hat{\boldsymbol{z}}\times\hat{\boldsymbol{y}} = -\hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{z}}\times\hat{\boldsymbol{z}} = 0$$



The resulting cross product expression is

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\,\hat{\boldsymbol{x}} + (A_z B_x - A_x B_z)\,\hat{\boldsymbol{y}} + (A_x B_y - A_y B_x)\,\hat{\boldsymbol{z}}$$

This cross product result can also be written compactly in the form of a determinant as

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

<u>Example</u> (Dot product / Cross product)

Given $\boldsymbol{E} = 3\hat{\boldsymbol{y}} + 4\hat{\boldsymbol{z}}$ and $\boldsymbol{F} = 4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}$, determine

(a.) the vector component of ***E*** in the direction of ***F***.

(b.) a unit vector perpendicular to both ***E*** and ***F***.

(a.) To find the vector component of ***E*** in the direction of ***F***, we must dot the vector ***E*** with the unit vector in the direction of ***F***.

$$\hat{\boldsymbol{a}}_F = \frac{\boldsymbol{F}}{|\boldsymbol{F}|} = \frac{4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}}{\sqrt{4^2 + 10^2 + 5^2}} = \frac{1}{\sqrt{141}}(4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}})$$

The dot product of ***E*** and $\boldsymbol{a}_F$ is

$$\boldsymbol{E} \cdot \hat{\boldsymbol{a}}_F = (3\hat{\boldsymbol{y}} + 4\hat{\boldsymbol{z}}) \cdot \frac{1}{\sqrt{141}}(4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}})$$

$$= \frac{1}{\sqrt{141}}[(3)(-10) + (4)(5)] = -\frac{10}{\sqrt{141}}$$

(Scalar component of ***E*** along ***F***)

The vector component of ***E*** along ***F*** is

$$(\boldsymbol{E} \cdot \hat{\boldsymbol{a}}_F)\hat{\boldsymbol{a}}_F = -\frac{10}{141}(4\hat{\boldsymbol{x}} - 10\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}})$$

(b.) To find a unit vector normal to both ***E*** and ***F***, we use the cross product. The result of the cross product is a vector which is normal to both ***E*** and ***F***.

$$\boldsymbol{E} \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}})$$

We then divide this vector by its magnitude to find the unit vector.

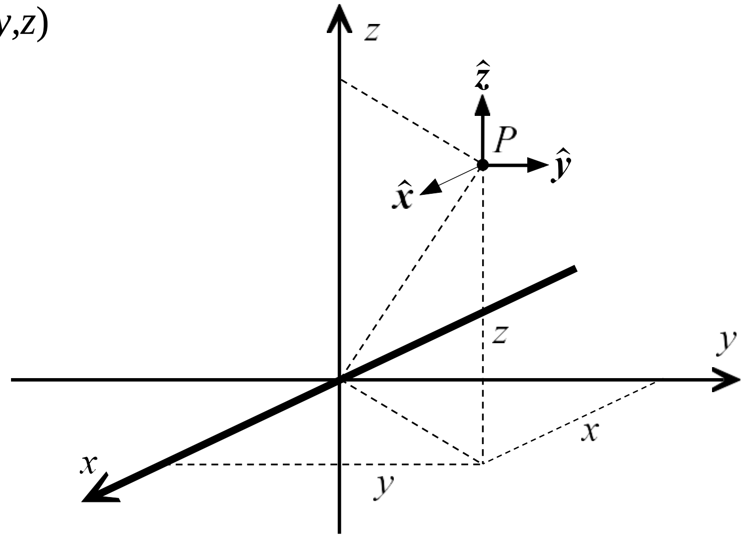
$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{E}\times\boldsymbol{F}}{|\boldsymbol{E}\times\boldsymbol{F}|} = \frac{55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}}}{\sqrt{55^2 + 16^2 + 12^2}} = \frac{1}{\sqrt{3425}}(55\hat{\boldsymbol{x}} + 16\hat{\boldsymbol{y}} - 12\hat{\boldsymbol{z}})$$

The negative of this unit vector is also normal to both ***E*** and ***F***.

Coordinate and Unit Vector Definitions

Rectangular Coordinates (x,y,z)

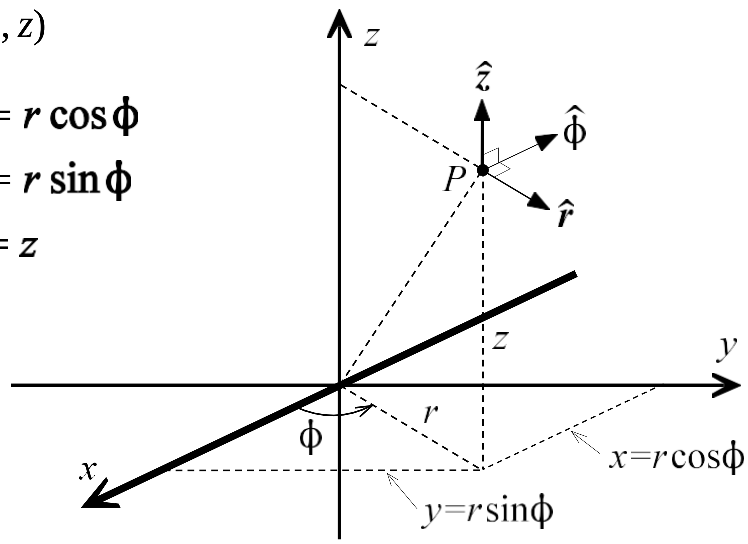
$$(-\infty < x < \infty)$$
$$(-\infty < y < \infty)$$
$$(-\infty < z < \infty)$$



Cylindrical Coordinates (r,ϕ,z)

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\phi$$
$$\phi = \tan^{-1}(y/x) \qquad y = r\sin\phi$$
$$z = z \qquad z = z$$

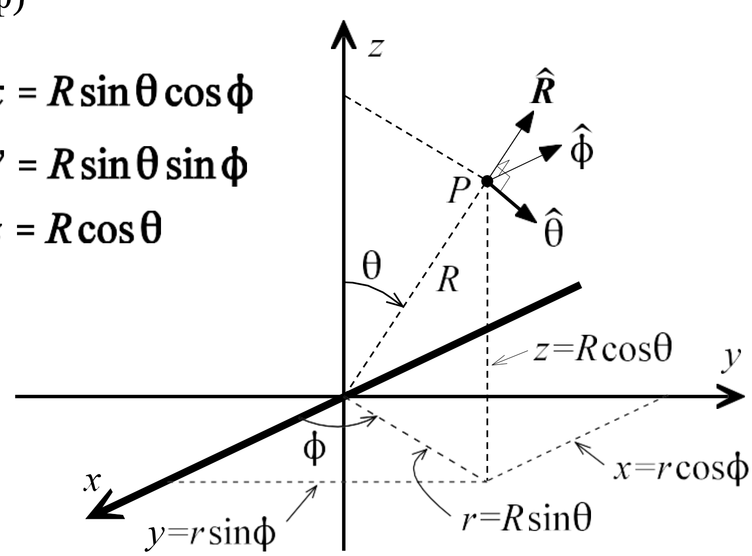
$$(0 \le r < \infty)$$
$$(0 \le \phi < 2\pi)$$
$$(-\infty < z < \infty)$$



Spherical Coordinates (R,θ,ϕ)

$$R = \sqrt{x^2+y^2+z^2} \qquad x = R\sin\theta\cos\phi$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad y = R\sin\theta\sin\phi$$
$$\phi = \tan^{-1}(y/x) \qquad z = R\cos\theta$$

$$(0 \le R < \infty)$$
$$(0 \le \theta \le \pi)$$
$$(0 \le \phi < 2\pi)$$



Vector Definitions and Coordinate Transformations

Vector Definitions

Rectangular $\boldsymbol{A} = A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}} = (A_x, A_y, A_z)$

Cylindrical $\boldsymbol{A} = A_r \hat{\boldsymbol{r}} + A_\phi \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} = (A_r, A_\phi, A_z)$

Spherical $\boldsymbol{A} = A_R \hat{\boldsymbol{R}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}} = (A_R, A_\theta, A_\phi)$

Vector Magnitudes

$$\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|\,|\boldsymbol{A}| \cos 0^o = |\boldsymbol{A}|^2 \quad \Rightarrow \quad |\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$$

Rectangular $|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Cylindrical $|\boldsymbol{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2}$

Spherical $|\boldsymbol{A}| = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$

Rectangular to Cylindrical Coordinate Transformation

$$(A_x, A_y, A_z) \longrightarrow (A_r, A_\phi, A_z)$$

The transformation of rectangular to cylindrical coordinates requires that we find the components of the rectangular coordinate vector $\boldsymbol{A}$ in the direction of the cylindrical coordinate unit vectors (using the dot product).
The required dot products are

$$A_r = \boldsymbol{A} \cdot \hat{\boldsymbol{r}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{r}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{r}}$$

$$A_\phi = \boldsymbol{A} \cdot \hat{\boldsymbol{\phi}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}}$$

$$A_z = \boldsymbol{A} \cdot \hat{\boldsymbol{z}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{z}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{z}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{z}} = A_z$$

where the $\hat{\boldsymbol{z}}$ unit vector is identical in both orthogonal coordinate systems such that

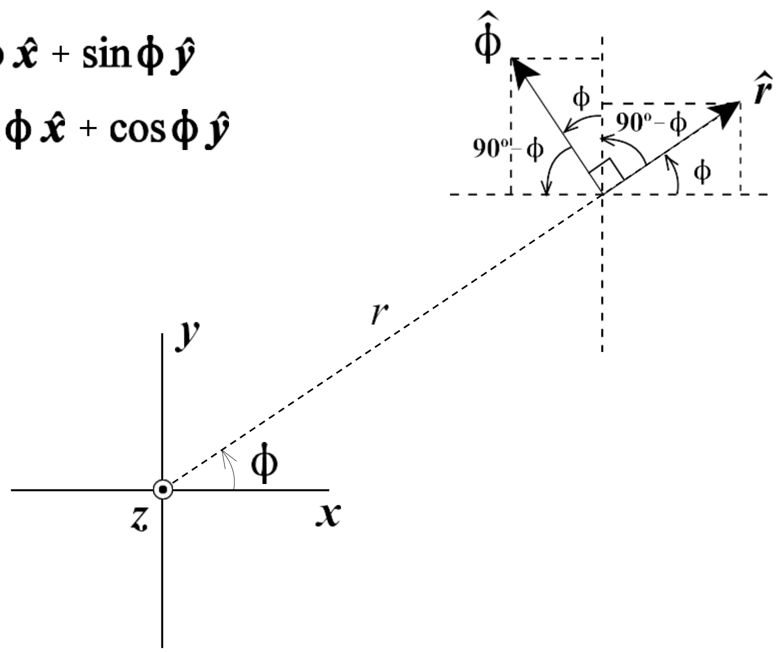
$$\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}} = 0 \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}} = 0$$

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{z}} = 0 \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{z}} = 0 \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{z}} = 1$$

The four remaining unit vector dot products are determined according to the geometry relationships between the two coordinate systems.

$$\hat{\boldsymbol{r}} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$$



$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\cdot(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}) = \cos\phi$$

$$\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{r}} = \hat{\boldsymbol{y}}\cdot(\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}) = \sin\phi$$

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{x}}\cdot(-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}) = -\sin\phi$$

$$\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{y}}\cdot(-\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}) = \cos\phi$$

The resulting cylindrical coordinate vector is

$$\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\boldsymbol{z}}$$

$$= (A_x\cos\phi + A_y\sin\phi)\hat{\boldsymbol{r}} + (A_y\cos\phi - A_x\sin\phi)\hat{\boldsymbol{\phi}} + A_z\hat{\boldsymbol{z}}$$

In matrix form, the rectangular to cylindrical transformation is

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

<u>Cylindrical to Rectangular Coordinate Transformation</u>

$(A_r, A_\phi, A_z) \longrightarrow (A_x, A_y, A_z)$

The transformation from cylindrical to rectangular coordinates can be determined as the inverse of the rectangular to cylindrical transformation.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

The cylindrical coordinate variables in the transformation matrix must be expressed in terms of rectangular coordinates.

$$\cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

The resulting transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

The cylindrical to rectangular transformation can be written as

$$\begin{aligned} \boldsymbol{A} &= A_x \hat{\boldsymbol{x}} + A_y \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}} \\ &= (A_r \cos\phi - A_\phi \sin\phi)\, \hat{\boldsymbol{x}} + (A_r \sin\phi + A_\phi \cos\phi)\, \hat{\boldsymbol{y}} + A_z \hat{\boldsymbol{z}} \\ &= \left(A_r \frac{x}{\sqrt{x^2+y^2}} - A_\phi \frac{y}{\sqrt{x^2+y^2}} \right) \hat{\boldsymbol{x}} \\ &\qquad + \left(A_r \frac{y}{\sqrt{x^2+y^2}} + A_\phi \frac{x}{\sqrt{x^2+y^2}} \right) \hat{\boldsymbol{y}} \\ &\qquad\qquad + A_z \hat{\boldsymbol{z}} \end{aligned}$$

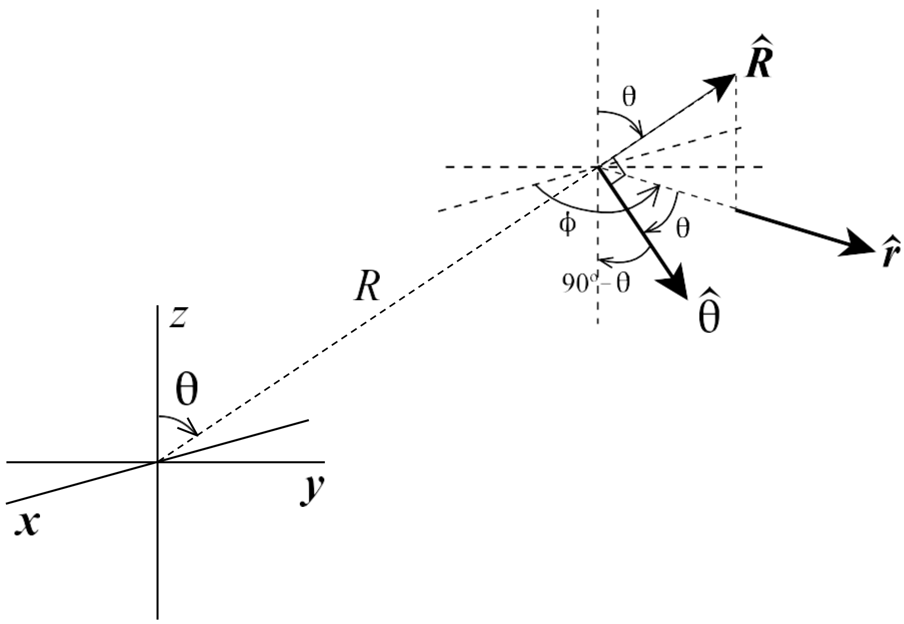
<u>Rectangular to Spherical Coordinate Transformation</u>

$(A_x, A_y, A_z) \;\rightarrow\; (A_R, A_\theta, A_\phi)$

The dot products necessary to determine the transformation from rectangular coordinates to spherical coordinates are

$$\begin{aligned} A_r &= \boldsymbol{A} \cdot \hat{\boldsymbol{R}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{R}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{R}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{R}} \\ A_\theta &= \boldsymbol{A} \cdot \hat{\boldsymbol{\theta}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\theta}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\theta}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} \\ A_\phi &= \boldsymbol{A} \cdot \hat{\boldsymbol{\phi}} = A_x \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\phi}} + A_y \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}} \end{aligned}$$

The geometry relationships between the rectangular and spherical unit vectors are illustrated below.



$$\hat{\boldsymbol{R}} = \sin\theta\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{z}}$$
$$= \sin\theta\,[\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}] + \cos\theta\,\hat{\boldsymbol{z}}$$
$$= \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{r}} - \cos(90^o - \theta)\,\hat{\boldsymbol{z}}$$
$$= \cos\theta\,[\cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}}] - \sin\theta\,\hat{\boldsymbol{z}}$$
$$= \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$$

The dot products are then

$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{R}} = \sin\theta\cos\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{R}} = \sin\theta\sin\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{R}} = \cos\theta$$
$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\theta}} = \cos\theta\sin\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\theta}} = -\sin\theta$$
$$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\phi}} = -\sin\phi \qquad \hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\phi}} = \cos\phi \qquad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}} = 0$$

and the rectangular to spherical transformation may be written as

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

<u>Spherical to Rectangular Coordinate Transformation</u>

$$(A_R, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$$

The spherical to rectangular coordinate transformation is the inverse of the rectangular to spherical coordinate transformation so that

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

The spherical coordinate variables in terms of the rectangular coordinate variables are

$$\sin\theta = \frac{r}{R} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \cos\theta = \frac{z}{R} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

The complete spherical to rectangular coordinate transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{-y}{\sqrt{x^2+y^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} & -\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$