Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

Subject Name: Electromagnatic Field

3th Class, Second Semester

Subject Code: [Insert Subject Code Here]

Academic Year: 2024-2025

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Lecture No.:- 4

Lecture Title: []

Coordinate Transformation Procedure

(1) Transform the component scalars into the new coordinate system.

(2) Insert the component scalars into the coordinate transformation matrix and evaluate.

Steps (1) and (2) can be performed in either order.

Example (Coordinate Transformations)

Given the rectangular coordinate vector

$$\boldsymbol{A} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\hat{\boldsymbol{x}} - \frac{yz}{\sqrt{x^2+y^2+z^2}}\,\hat{\boldsymbol{z}}$$

(a.) transform the vector $\boldsymbol{A}$ into cylindrical and spherical coordinates.

(b.) transform the rectangular coordinate point P (1,3,5) into cylindrical and spherical coordinates.

(c.) evaluate the vector $\boldsymbol{A}$ at P in rectangular, cylindrical and spherical coordinates.

(a.)

$$x = r\cos\phi \qquad A_x = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{r}{\sqrt{r^2+z^2}}$$

$$y = r\sin\phi$$

$$z = z \qquad A_z = -\frac{yz}{\sqrt{x^2+y^2+z^2}} = -\frac{zr\sin\phi}{\sqrt{r^2+z^2}}$$

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{r}{\sqrt{r^2+z^2}} \\ 0 \\ -\dfrac{zr\sin\phi}{\sqrt{r^2+z^2}} \end{bmatrix}$$

$$A_r = \frac{r\cos\phi}{\sqrt{r^2+z^2}} \qquad A_\phi = -\frac{r\sin\phi}{\sqrt{r^2+z^2}} = \qquad A_z = -\frac{zr\sin\phi}{\sqrt{r^2+z^2}}$$

$$\boldsymbol{A} = \frac{r}{\sqrt{r^2+z^2}}\left(\cos\phi\,\hat{\boldsymbol{r}} - \sin\phi\,\hat{\boldsymbol{\phi}} - z\sin\phi\,\hat{\boldsymbol{z}}\right)$$

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}$$

$$A_x = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{R\sin\theta}{R} = \sin\theta$$

$$A_z = -\frac{yz}{\sqrt{x^2+y^2+z^2}} = -\frac{R^2\sin\theta\cos\theta\sin\phi}{R}$$

$$= -R\sin\theta\cos\theta\sin\phi$$

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \\ -R\sin\theta\cos\theta\sin\phi \end{bmatrix}$$

$$\begin{aligned} A_R &= \sin^2\theta\cos\phi - R\sin\theta\cos^2\theta\sin\phi \\ A_\theta &= \sin\theta\cos\theta\cos\phi + R\sin^2\theta\cos\theta\sin\phi \\ A_\phi &= -\sin\theta\sin\phi \end{aligned}$$

$$\begin{aligned} \boldsymbol{A} = {} & \sin\theta\,(\sin\theta\cos\phi - R\cos^2\theta\sin\phi)\,\hat{\boldsymbol{R}} \\ & + \sin\theta\cos\theta\,(\cos\phi + R\sin\theta\sin\phi)\,\hat{\theta} \\ & - \sin\theta\sin\phi\,\hat{\boldsymbol{\phi}} \end{aligned}$$

(b.) $P\ (1, 3, 5) \quad \rightarrow \quad x = 1, y = 3, z = 5$

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 3^2} = \sqrt{10} = 3.16$$
$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$
$$z = z = 5$$

$$R = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35} = 5.92$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{1^2 + 3^2}}{5}\right) = 32.3^o$$
$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/1) = 71.6^o$$

$$\boxed{\mathrm{P}\ (1, 3, 5) \quad \rightarrow \quad P\ (3.16,\ 71.6^o,\ 5) \quad \rightarrow \quad P\ (5.92,\ 32.3^o,\ 71.6\ ^o)}$$

(c.)

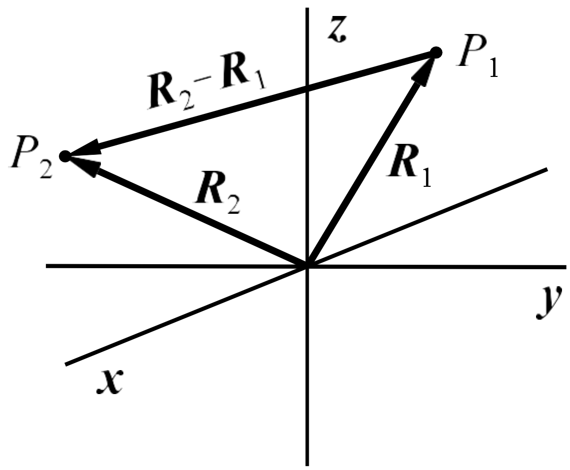
$$\boldsymbol{A}(1,3,5) = \frac{\sqrt{1^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}}\,\hat{\boldsymbol{x}} - \frac{(3)(5)}{\sqrt{1^2 + 3^2 + 5^2}}\,\hat{\boldsymbol{z}} = \boxed{0.535\,\hat{\boldsymbol{x}} - 2.54\,\hat{\boldsymbol{z}}}$$

$$\boldsymbol{A}(3.16, 71.6^o, 5) = \frac{3.16}{\sqrt{3.16^2 + 5^2}}\,(\cos 71.6^o\,\hat{\boldsymbol{r}} - \sin 71.6^o\,\hat{\boldsymbol{\phi}}$$
$$- 5 \sin 71.6^o\,\hat{\boldsymbol{z}})$$
$$= \boxed{0.169\,\hat{\boldsymbol{r}} - 0.507\,\hat{\boldsymbol{\phi}} - 2.53\,\hat{\boldsymbol{z}}}$$

$$\boldsymbol{A}(5.92, 32.3^o, 71.6^o) =$$
$$\sin 32.3^o\,(\sin 32.3^o \cos 71.6^o - 5.92 \cos^2 32.3^o \sin 71.6^o)\,\hat{\boldsymbol{R}}$$
$$+ \sin 32.3^o \cos 32.3^o\,(\cos 71.6^o + 5.92 \sin 32.3^o \sin 71.6^o)\,\hat{\boldsymbol{\theta}}$$
$$- \sin 32.3^o \sin 71.6^o\,\hat{\boldsymbol{\phi}}$$
$$= \boxed{-2.05\,\hat{\boldsymbol{R}} + 1.50\,\hat{\boldsymbol{\theta}} + 0.507\,\hat{\boldsymbol{\phi}}}$$

Separation Distances

Given a vector $\boldsymbol{R}_1$ locating the point P_1 and a vector $\boldsymbol{R}_2$ locating the point P_2, the distance d between the points is found by determining the magnitude of the vector pointing from P_1 to P_2, or vice versa.



$$d = |\boldsymbol{R}_2 - \boldsymbol{R}_1| = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$$

<u>Rectangular</u>

$$P_1 = (x_1, y_1, z_1) \qquad P_2 = (x_2, y_2, z_2)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

<u>Cylindrical</u>

$$P_1 = (r_1, \phi_1, z_1) \qquad P_2 = (r_2, \phi_2, z_2)$$

$$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

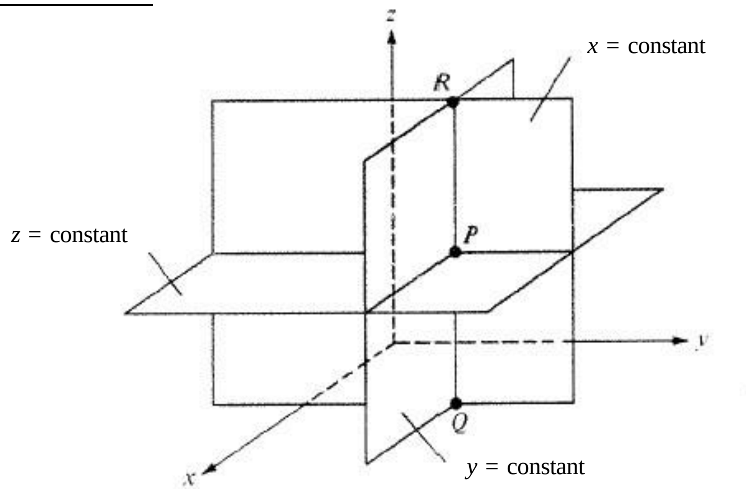
<u>Spherical</u>

$$P_1 = (R_1, \theta_1, \phi_1) \qquad P_2 = (R_2, \theta_2, \phi_2)$$

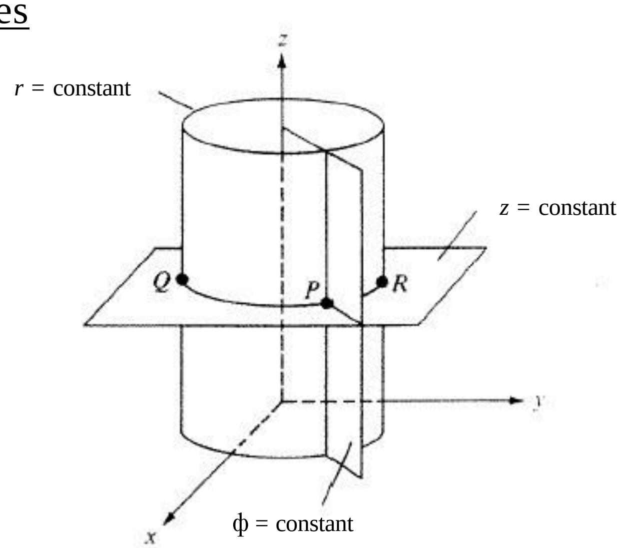
$$d = \sqrt{R_2^2 + R_1^2 - 2 R_1 R_2 \cos\theta_2 \cos\theta_1 - 2 R_1 R_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)}$$

Constant Coordinate Surfaces

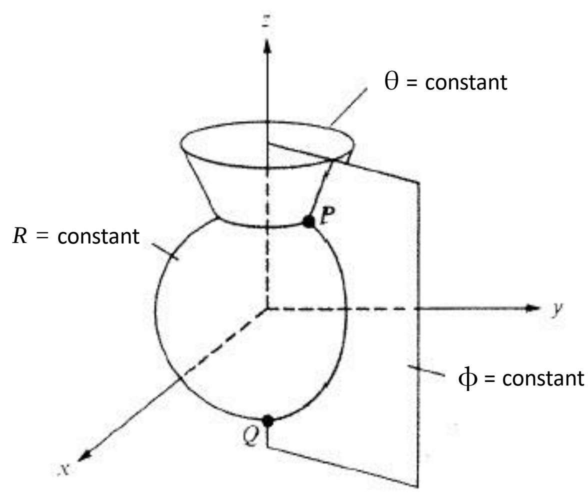
Rectangular Coordinates



Cylindrical Coordinates



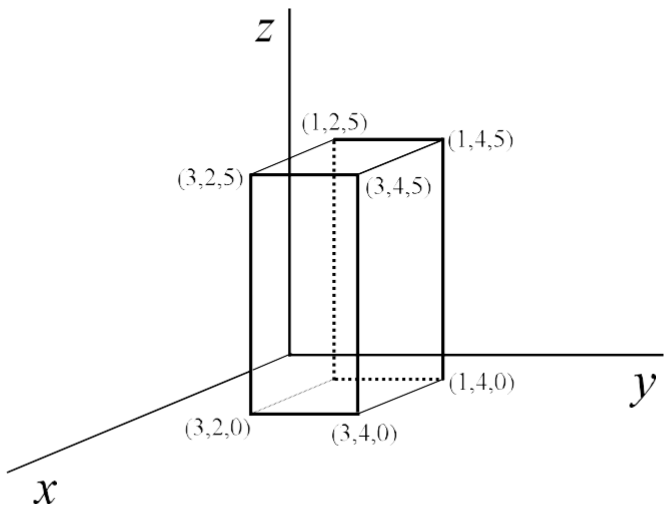
Spherical Coordinates



Volumes, Surfaces and Lines in Rectangular, Cylindrical and Spherical Coordinates

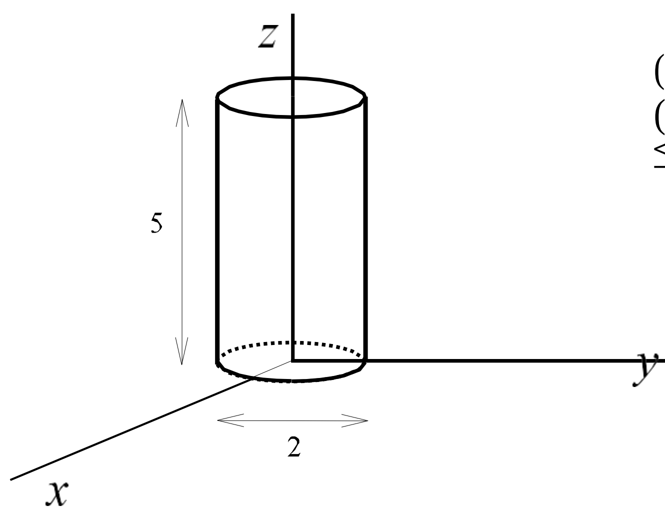
We may define particular volumes, surfaces and lines in rectangular, cylindrical and spherical coordinates by specifying ranges on the coordinate variables.

<u>Rectangular volume</u> (2×2×5 box)



$(1 \leq x \leq 3)$
$(2 \leq y \leq 4)$
$(0 \leq z \leq 5)$

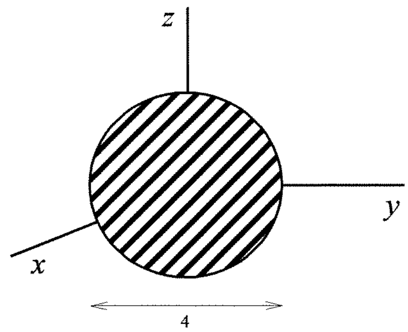
<u>Cylindrical volume</u> (cylinder of length = 5, diameter = 2)



$(0 \leq r \leq 1)$
$(0 \leq \phi \leq 2v)$ $(0 \leq z \leq 5)$

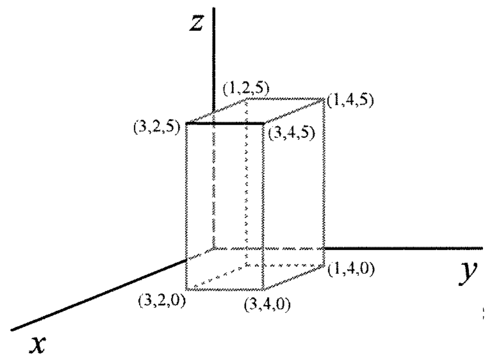
Surface on the Spherical volume
(outer surface of the sphere)

$R = 2$ $\quad\quad$ (r – constant)
$(0 \le \theta \le \pi)$
$(0 \le \phi \le 2\pi)$



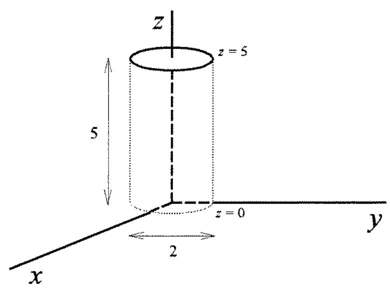
Line on the Rectangular volume
(upper edge of the front face)

$x = 3$ $\quad\quad$ (x – constant)
$(2 \le y \le 4)$
$z = 5$ $\quad\quad$ (z – constant)



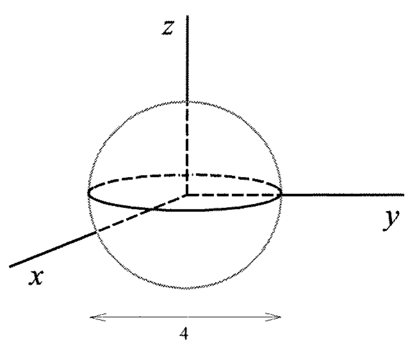
Line on the Cylindrical volume
(outer edge of the upper surface)

$r = 1$ $\quad\quad$ (r – constant)
$(0 \le \phi \le 2\pi)$
$z = 5$ $\quad\quad$ (z – constant)



Line on the Spherical volume
(equator of the sphere)

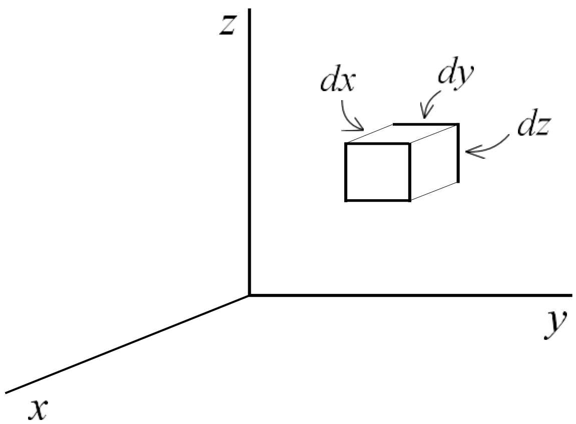
$R = 2$ $\quad\quad$ (R – constant)
$\theta = \pi/2$ $\quad\quad$ (θ – constant)
$(0 \le \phi \le 2\pi)$



Differential Lengths, Surfaces and Volumes

When integrating along lines, over surfaces, or throughout volumes, the ranges of the respective variables define the limits of the respective integrations. In order to evaluate these integrals, we must properly define the differential elements of length, surface and volume in the coordinate system of interest. The definition of the proper differential elements of length (*dl* for line integrals) and area (*ds* for surface integrals) can be determined directly from the definition of the differential volume (*dv* for volume integrals) in a particular coordinate system.

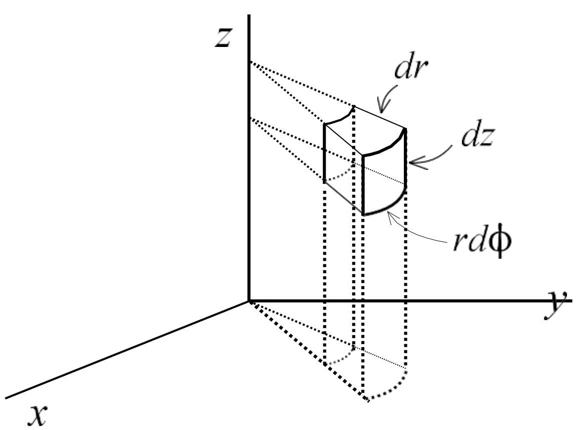
<u>Rectangular Coordinates</u>



$$dv = (dx)(dy)(dz)$$

x−constant	$ds = dy\,dz$
y−constant	$ds = dx\,dz$
z−constant	$ds = dx\,dy$
x,y−constant	$dl = dz$
x,z−constant	$dl = dy$
y,z−constant	$dl = dx$

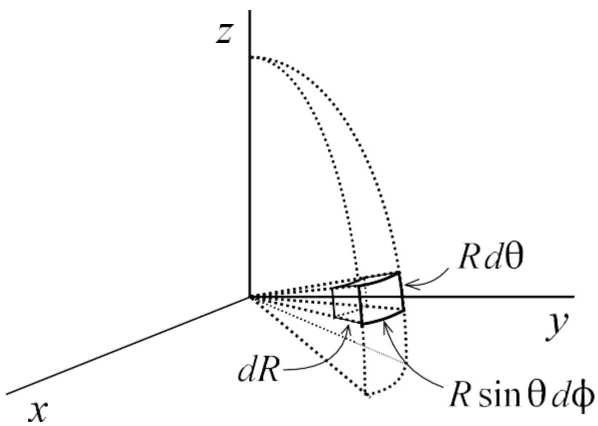
<u>Cylindrical Coordinates</u>



$$dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz$$

r−constant	$ds = r_o\,d\phi\,dz$
ϕ−constant	$ds = dr\,dz$
z−constant	$ds = r\,dr\,d\phi$
r,ϕ−constant	$dl = dz$
r,z−constant	$dl = r_o d\phi$
ϕ,z−constant	$dl = dr$

Spherical Coordinates



$$dv = (dR)(R\,d\theta)(R\sin\theta\,d\phi)$$
$$= R^2\sin\theta\,dR\,d\theta\,d\phi$$

R − constant $\quad ds = R_o^2\sin\theta\,d\theta\,d\phi$

θ − constant $\quad ds = R\sin\theta_o\,dR\,d\phi$

ϕ − constant $\quad ds = R\,dR\,d\theta$

R,θ − constant $\quad dl = R_o\sin\theta_o\,d\phi$

R,ϕ − constant $\quad dl = R_o\,d\theta$

θ,ϕ − constant $\quad dl = dR$

<u>Example</u> (Line/surface/volume integration)

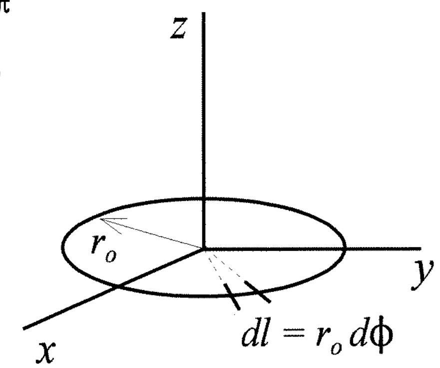
Using the appropriate differential elements, show that

(a.) the circumference of a circle of radius r_o is $2\pi r_o$.
(b.) the surface area of a sphere of radius R_o is $4\pi R_o^2$.
(c.) the volume of a sphere of radius R_o is $(4/3)\pi R_o^3$.

(a.)

$$L = \int dl = \int_0^{2\pi} r_o d\phi = r_o \int_0^{2\pi} d\phi = r_o [\phi] \Big|_0^{2\pi}$$

$$\boxed{L = 2\pi r_o}$$



(b.)

$$S = \int\int ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} R_o^2 \sin\theta \, d\theta \, d\phi$$

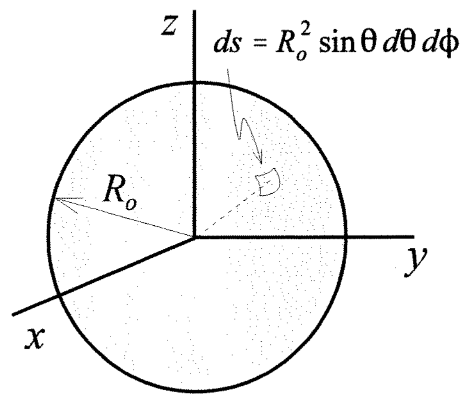
$$= R_o^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= R_o^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= R_o^2 \, [-\cos\theta] \Big|_0^{\pi} [\phi] \Big|_0^{2\pi}$$

$$= R_o^2 (2)(2\pi)$$

$$\boxed{S = 4\pi R_o^2}$$



(c.)

$$V = \iiint dv$$

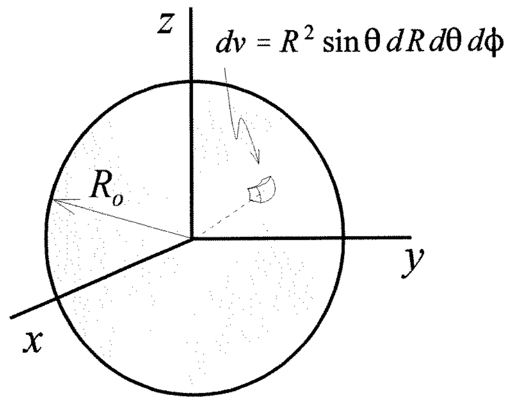
$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{R_o} R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \int_0^{R_o} R^2 dR \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{R^3}{3}\right]\Bigg|_0^{R_o} [-\cos\theta]\Big|_0^{\pi} [\phi]\Big|_0^{2\pi}$$

$$= \left(\frac{R_o^3}{3}\right)(2)(2\pi)$$

$$\boxed{V = \frac{4}{3}\pi R_o^3}$$



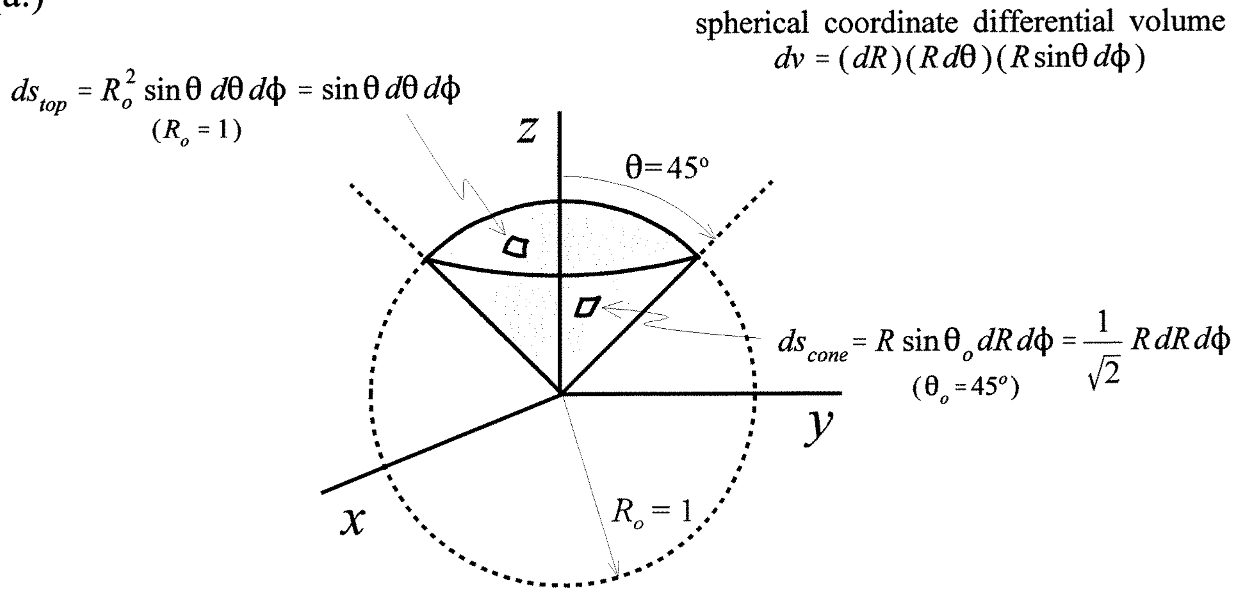
Example (Surface/volume integration in spherical coordinates)

A three-dimensional solid is described in spherical coordinates according to

$$(0 \le R \le 1) \qquad (0 \le \theta \le \pi/4) \qquad (0 \le \phi \le 2\pi)$$

(a.) Sketch the solid.
(b.) Determine the volume of the solid.
(c.) Determine the surface area of the solid

(a.)



(b.)

$$V = \iiint dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{R=0}^{1} R^2 \sin\theta\, dR\, d\theta\, d\phi$$

$$= \int_0^1 R^2 dR \int_{\theta=0}^{\pi/4} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi = \left[\frac{R^3}{3}\right]\Bigg|_0^1 \left[-\cos\theta\right]\Big|_0^{\pi/4} \left[\phi\right]\Big|_0^{2\pi}$$

$$= \left(\frac{1}{3}\right)\left(-\frac{1}{\sqrt{2}} + 1\right)(2\pi)$$

$$\boxed{V = 0.613 \text{ units}^3}$$

(c.)

$$S = S_{top} + S_{cone}$$

$$= \iint_{top\ surface} ds_{top} + \iint_{cone\ surface} ds_{cone}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \sin\theta \, d\theta \, d\phi + \frac{1}{\sqrt{2}} \int_{\phi=0}^{2\pi} \int_{R=0}^{1} R \, dR \, d\phi$$

$$= \int_{0}^{\pi/4} \sin\theta \, d\theta \int_{0}^{2\pi} d\phi + \frac{1}{\sqrt{2}} \int_{0}^{1} R \, dR \int_{0}^{2\pi} d\phi$$

$$= [-\cos\theta]\Big|_{0}^{\pi/4} [\phi]\Big|_{0}^{2\pi} + \frac{1}{\sqrt{2}} \left[\frac{R^2}{2}\right]\Bigg|_{0}^{1} [\phi]\Big|_{0}^{2\pi}$$

$$= \left[-\frac{1}{\sqrt{2}} + 1\right](2\pi) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)(2\pi)$$

$$S = 4.06 \ \text{units}^2$$