Chapter Three

Torsion

When regular circular shaft is subjected to torque , every cross-section in the shaft will be subjected to pure shear condition and the resisting torque which is produced from shearing stress will be equal in magnitude and having opposite direction to the using torque in deriving the torsion formula .

For the purposes of deriving a simple theory to describe the behavior of shaft that subjected to torque , it is necessary to make the following basic assumptions :

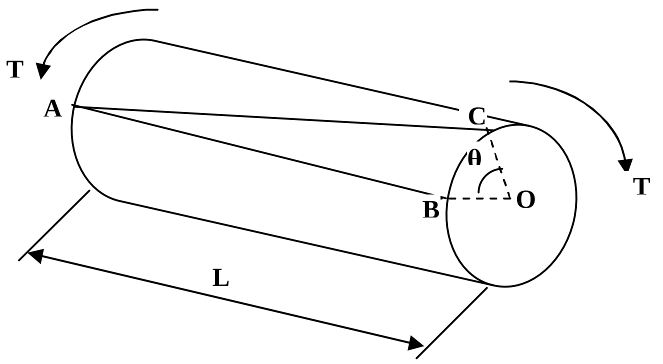
1- Circular section remain circular .

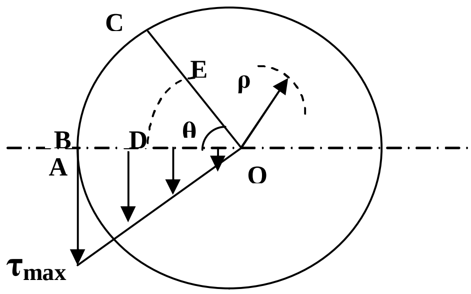
2- Cross-section remain plane (this is certainly not the case with the torsion of non circular section).

3- Material is elastic and obey to Hook's law (shear stress is proportional linearly with shear strain).

4- Material is homogenous and having equally elasticity properties .

5- The cross-section rotate as rigid , i.e. every diameter rotates with same angle .





If the torque (T) is applied at the free end of the shaft , the fiber (AB) on outside surface will be twisted into (AC) as the shaft is twisted through the angle (θ) .

Consider any internal fiber located a radial distance (ρ) from the axis (Centre of the shaft) , the radius will also rotates through angle (θ) causing total shearing deformation (δ_s) equal to arc length (DE).

$$\delta_s = DE = \rho\theta \qquad \text{---------(1)}$$

as shearing strain $$(\gamma) = \frac{\delta_s}{L} = \frac{\rho\theta}{L} \qquad \text{---------(2)}$$

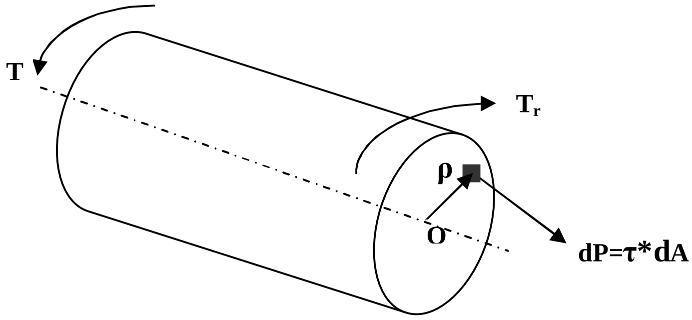
From Hook's law , shearing stress at this fiber is (τ)

$$\tau = G * \gamma = \frac{G\theta}{L}\rho \qquad \text{---------(3)}$$

where :

G------- Modulus of rigidity (N/m^2)

The above equation shows that the stress distribution along any radius varies linearly with the radius distance from the axis of the shaft . The maximum shear stress (τ_{max}) occur at the outside fiber .



Now consider the shaft in figure :

The differential area (dA) at radial distance (ρ) carries the differential resisting load (dP).

$$dP = \tau * dA \quad \text{---------(4)}$$

and the stress is to be assumed uniform over each area .

From static equilibrium :

$$\sum M_O = 0 \;, \; T = T_r$$

where : T_r**---------resisting torque ,** T**---------applied torque**

developed by all differential loads (dP)

$$T = T_r = \int \rho * dP = \int \rho * \tau * dA \quad \text{-----------(5)}$$

sub (τ) by the value in equation (3)

$$\therefore T = \frac{G\theta}{L} \int \rho^2 dA \quad \text{------------(6)}$$

Since the term $\left(\int \rho^2 dA\right) = J$ **(polar moment of inertia)**

$$\therefore T = \frac{G\theta}{L} J \quad \text{-------------(7)}$$

$$\therefore \theta = \frac{T * L}{G * J} \quad \text{-------------(8)}$$

where :

θ(radians) , T(N.m) , L(m) , J(m^4) , G(N/m^2)

<u>Note</u> : If (θ) in degrees , the right hand side is multiple by term $\left(\frac{180}{\pi}\right)\frac{\deg}{rad}$

or

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{\rho}$$

$$\therefore \tau = \frac{T * \rho}{J} \qquad \text{--------------(9)}$$

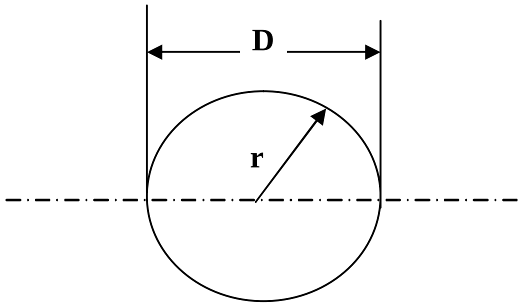
As the maximum stress occur at the surface of shaft when (*ρ*=*r*)

For maximum shear stress :

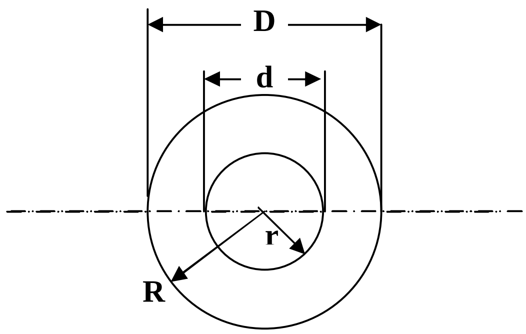
$$\tau_{max.} = \frac{T * r}{J} \qquad \text{-------------(10)}$$

To find the polar moment of inertia for solid and hollow shafts we used the following equations:

$$J = \int_0^R 2\pi * r^3 dr$$ **General form for polar moment of inertia**



$$J = \frac{\pi r^4}{2} = \frac{\pi D^4}{32}$$ **(For solid shaft)**



$$J = \frac{\pi}{2}\left(R^4 - r^4\right) = \frac{\pi}{32}\left(D^4 - d^4\right)$$ **(For hollow shaft)**

Power Transmitted by Shaft :

In many practical applications , shafts are used to transmit power from dynamics , its is know that the power (p) transmitted by a constant torque (T) rotating at constant angular speed (ω) is given by :-

$$p = T * \omega \qquad \text{----------(11)}$$

where (ω) measured in radians per unit time

$$\omega = 2\pi * f \qquad \text{----------(12)}$$

where (*f*) is a frequency of rotation

$$\therefore\ p = T * 2\pi * f \qquad \text{----------(13)}$$

If the power given by horse power (h.p)

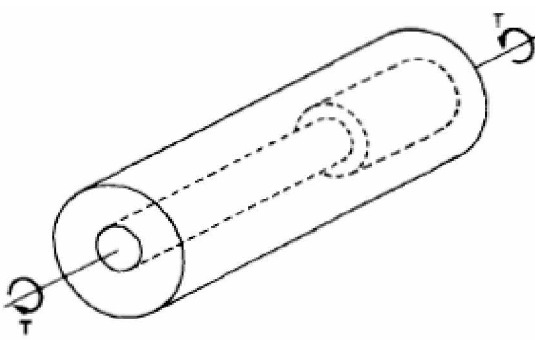
[h.p* 746 or (750)= watt]

Shafts Connection :

1- Series Connection :

If two or more shafts of different material , diameter or basic form are connected together in such way that each carries the same torque , then the shafts are said to be connected in *series* and the composite shaft so produce is therefore termed series-connected as shown in figure .

In such cases the composite shaft strength is treated by considering each composite shaft separately , applying the torsion theory to each in turn , the composite shaft will therefore be as weak as it weakest component .



Composite shaft (Series connection)

If relative dimensions of various parts are required then a solution is usually effected by equating the torques in each shaft .

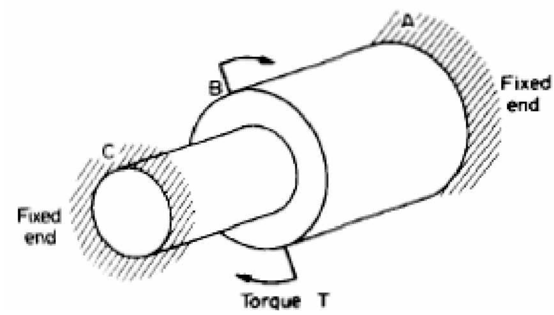
$$T = \frac{G_1 * J_1 * \theta_1}{L_1} = \frac{G_2 * J_2 * \theta_2}{L_2}$$

n $\theta_1 = \theta_2$ **, and similar shaft material (G_1=G_2)**

$$\therefore \frac{L_1}{J_1} = \frac{L_2}{J_2}$$

2- Parallel Connection :

If two or more materials are rigidly fixed together in such way that the applied torque is shared between them, then the composite shaft so formed is said to be connected in parallel connection .



Composite shaft (Parallel connection)

$$\theta_t = \theta_1 + \theta_2$$
$$T_t = T_1 + T_2$$

For two fixed ends shaft $(\theta_1 + \theta_2 = 0)$

$$\therefore \theta_1 = -\theta_2$$

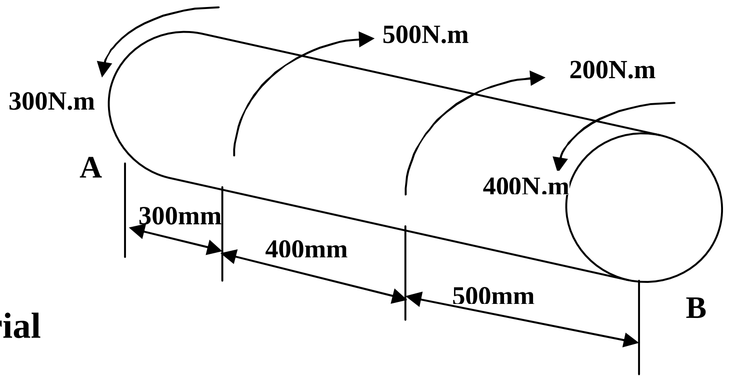
$$\therefore \frac{T_1 * L_1}{G_1 * J_1} = -\frac{T_2 * L_2}{G_2 * J_2}$$

For equally length shaft : $\therefore \frac{T_1}{T_2} = \frac{G_1 * J_1}{G_2 * J_2}$

$$\tau_{1_{MAX}} = \frac{T_1 * R_1}{J_1} \quad , \quad \tau_{2_{MAX}} = \frac{T_2 * R_2}{J_2}$$

Ex-1- A steel shaft shown in figure (d=40mm) is subjected to the torques as shown, determine the angle of twist of point (B) with respect to point (A) in degrees .

G_{st} =75GPa .



Sol :-

$$\theta = \sum \frac{T*L}{J*G}$$

For the same diameter and same material

$$\Rightarrow \quad (J \;,\; G \;) \text{ are constant}$$

$$\therefore \theta = \frac{1}{G*J}\left[T_1*L_1 + T_2*L_2 + T_3*L_3\right]$$

$$\therefore \theta_{B/A} = \frac{1}{\frac{\pi}{32}(0.04)^2 * 75*10^9}\left[400*0.5 + 200*0.4 - 300*0.3\right]$$

$$\therefore \theta = 0.01 \text{ radians}$$

in degrees $\Rightarrow 0.01*\frac{180}{\pi} = 0.578$ **degree**

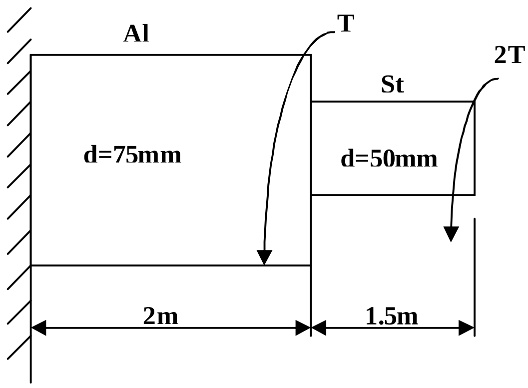
Ex-2- A compound shaft consisting of an aluminum segment and steel segment , is acted upon by two torques as shown in figure . determine the maximum permissible value of (T) subjected to the following conditions :

1) maximum shear stress in steel is (100MPa) .

2) maximum shear stress in aluminum is (70MPa) .

3) the angle of twist of free end is limited to (12^o) . (G_{st}=83GPa , G_{Al} =28GPa) .

Sol :- According to shear stress



1- For steel shaft :

$$\tau_{max.} = \frac{T*r}{J} = \frac{16T}{\pi * d^3}$$

$$\therefore 100*10^6 = \frac{16(2T)}{\pi(0.05)^3}$$

$$T = 1.23kN.m$$

2- For aluminum shaft :

$$\tau_{max.} = \frac{T*r}{J} = \frac{16T}{\pi * d^3}$$

$$\therefore 70*10^6 = \frac{16(3T)}{\pi(0.075)^3}$$

$$T = 1.93kN.m$$

According to angle of twist :

$$\theta = \frac{T * L}{G * J} \qquad \Rightarrow \theta = \frac{T_{St} * L_{St}}{G_{St} * J_{St}} + \frac{T_{Al} * L_{Al}}{G_{Al} * J_{Al}}$$

$$\text{n } \theta_{total} = \theta_{St} + \theta_{Al}$$

$$\therefore 12 * \frac{\pi}{180} = \frac{2T * 1.5}{\frac{\pi}{32}(0.05)^4 * 83 * 10^9} + \frac{3T * 2}{\frac{\pi}{32}(0.075)^4 * 28 * 10^9} \Rightarrow T = 1.638\,kN.m$$

$\therefore$ *Maximum* (T) *is* $(1.23\,kN.m)$ **are can be used**

Flange Bolt Couplings :

A flange bolt coupling is a connection commonly used between two shafts . It consist of flanges rigidly attached to the ends of the shafts and bolted together .

(P) is the shear force created in the bolts which transmits the torque .

The torque resistance of one bolt is :

T=P*R

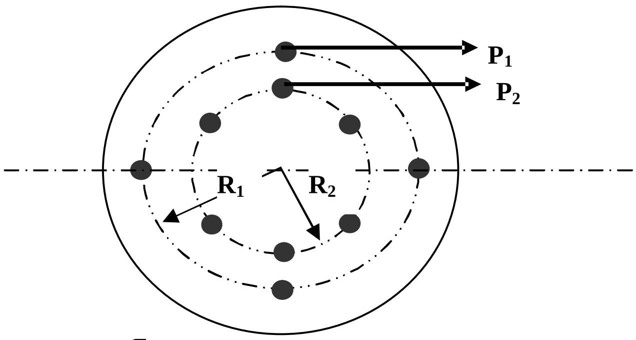
where :

R---------radius of the bolts circle for (n) number of the bolts .

$$\therefore T = P * R * n = \frac{\pi}{4} d^2 * \tau * R * n$$

If the coupling has two concentric rows of bolts as shown in figure

$$\therefore \quad T = P_1 * R_1 * n_1 + P_2 * R_2 * n_2$$



The shearing strains are related by :

$$\frac{\gamma_1}{R_1} = \frac{\gamma_2}{R_2}$$

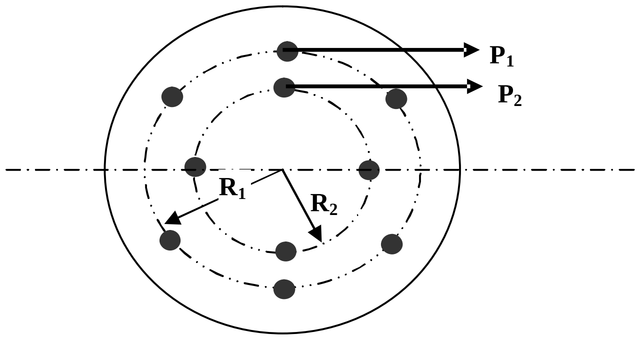
Using Hook's law for shear : $G = \frac{\tau}{\gamma} \Rightarrow \gamma = \frac{\tau}{G}$

$$\therefore \frac{\tau_1}{G_1 * R_1} = \frac{\tau_2}{G_2 * R_2} \quad \textbf{or} \quad \frac{\frac{P_1}{A_1}}{G_1 * R_1} = \frac{\frac{P_2}{A_2}}{G_2 * R_2}$$

If the bolts in two circles have the same area ($A_1=A_2$) and are made from same materials ($G_1=G_2$) $\Rightarrow$ $\therefore \frac{P_1}{R_1} = \frac{P_2}{R_2}$

Ex-3- A flange bolt coupling consist of six (10mm) stead bolt on bolt circle of (300mm) in diameter and four (10mm) diameter on circle of bolt of (200mm) . Determine the torque can be applied without exceeding (60MPa) in shear for bolt .

Sol :-



$$\frac{P_1}{R_1} = \frac{P_2}{R_2}$$

$$\frac{P_1}{150} = \frac{P_2}{100} \Rightarrow P_2 = \frac{2}{3}P_1$$

$$T = P_1 * R_1 * n_1 + P_2 * R_2 * n_2$$

$$T = P_1\left[R_1 * n_1 + \frac{2}{3}R_2 * n_2\right]$$

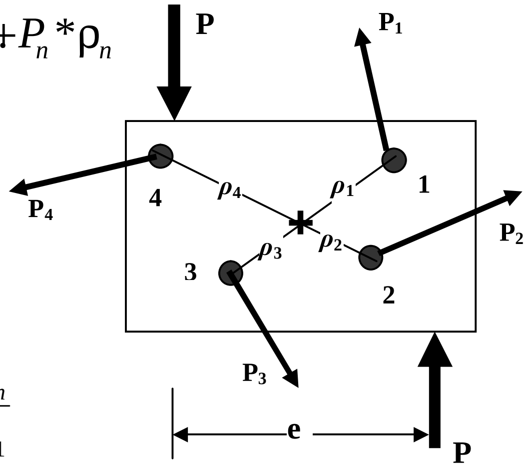
$$\tau = \frac{P}{A} \Rightarrow P = \tau * A$$

$$\therefore T_{\max.} = \frac{\pi}{4}(0.01)^2 * 60 * 10^6 * \left[0.15 * 6 + \frac{2}{3} * 0.1 * 4\right]$$

$$\therefore T_{\max.} = 5.5kN.m$$

When the bolts (or rivets) distributed in non-circular shape , the rotational moment will be about the centre of bolts (or rivets) distribution .

$$P*e = P_1*\rho_1 + P_2*\rho_2 + P_3*\rho_3 + P_4*\rho_4 + ... + P_n*\rho_n$$



P_i **is proportional to** ρ_i

$$\therefore \frac{P_1}{\rho_1} = \frac{P_2}{\rho_2} = \frac{P_3}{\rho_3} = \frac{P_4}{\rho_4} = ... \frac{P_n}{\rho_n}$$

$$\therefore P_2 = P_1\frac{\rho_2}{\rho_1} \;\; , \;\; P_3 = P_1\frac{\rho_3}{\rho_1} \;\; , \;\; P_4 = P_1\frac{\rho_4}{\rho_1} \;\; , \;\; P_n = P_1\frac{\rho_n}{\rho_1}$$

$$\therefore P*e = P_1*\rho_1 + P_1\frac{\rho_2}{\rho_1}\rho_2 + P_1\frac{\rho_3}{\rho_1}\rho_3 + P_1\frac{\rho_4}{\rho_1}\rho_4 + ...+ P_1\frac{\rho_n}{\rho_1}\rho_n$$

$$\therefore P_1 = \frac{P*e*\rho_1}{\rho^2{}_1 + \rho^2{}_2 + \rho^2{}_3 + \rho^2{}_4 + ...+ \rho^2{}_n}$$

similarly :- $$P_2 = \frac{P*e*\rho_2}{\rho^2{}_1 + \rho^2{}_2 + \rho^2{}_3 + \rho^2{}_4 + ...+ \rho^2{}_n}$$

$$P_3 = \frac{P*e*\rho_3}{\rho^2{}_1 + \rho^2{}_2 + \rho^2{}_3 + \rho^2{}_4 + ...+ \rho^2{}_n}$$

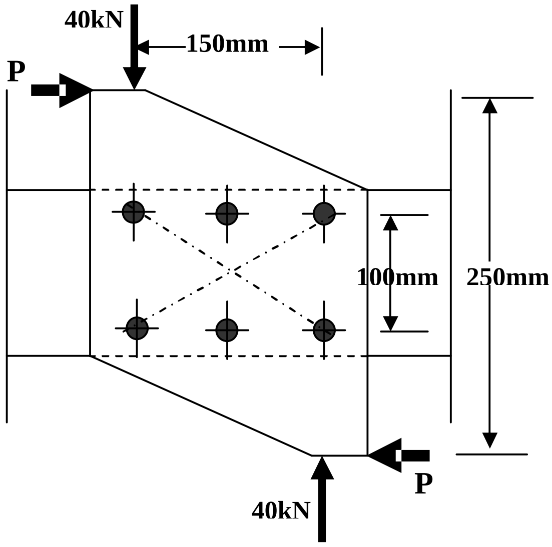
$$P_4 = \frac{P*e*\rho_4}{\rho^2{}_1 + \rho^2{}_2 + \rho^2{}_3 + \rho^2{}_4 + ...+ \rho^2{}_n}$$

Generally :

$$P_i = \frac{P*e*\rho_i}{\rho^2{}_1 + \rho^2{}_2 + \rho^2{}_3 + \rho^2{}_4 + ...+ \rho^2{}_n} = \frac{P*e*\rho_i}{\sum_{k=1}^{n}\rho_k^2}$$

$$\therefore \tau_i = \frac{(P*e)\rho_i}{\sum_{k=1}^{n} A*\rho_k^2}$$

Ex-4- Six (20mm) diameter rivets fasten the plate as shown in figure to the fixed member . Determine the average shearing stress caused in each rivet by (40kN) loads . what additional loads (P) can be applied before the shearing stress in any rivet exceeds (60MPa) .



Sol:- 1)

$$\tau_i = \frac{(P*\rho)*\rho_i}{\sum_{k=1}^{n} A*\rho_k^2}$$

$$\rho = \sqrt{(75)^2+(50)^2} = 90mm$$

$$T = P*\rho$$

$$\therefore \tau = \frac{6*10^3*0.09}{\frac{\pi}{4}(0.02)^2\left[2(0.05)^2+4(0.09)^2\right]}$$

$\therefore \tau = 45\ MPa$ **at the large distance rivet**

$$\therefore \tau = \frac{6*10^3*0.05}{\frac{\pi}{4}(0.02)^2\left[2(0.05)^2+4(0.09)^2\right]} = 25MPa$$ **at the small distance rivets .**

2)

$$60*10^6 = \frac{T_2*0.09}{\frac{\pi}{4}(0.02)^2\left[2(0.05)^2+4(0.09)^2\right]}$$

$$T_2 = 7.83kN.m$$

$$\therefore T = 6+T_2 \Rightarrow T = 6+7.83 = 13.84kN.m$$

$$P = \frac{T}{\rho_2} \Rightarrow P = \frac{13.84*10^3}{0.25} = 55.36kN$$

Helical Spring :-

The spring is composed of a wire or round rod of diameter (d) wound into a helix of mean radius (R) .

In figure (b) to balance the applied axial load (P) , the exposed shaded cross-section of the spring must provide the resistance (P_r) equal to (P).

(P) and (P_r) equal and opposite and parallel crea a couple of magnitude (PR) which must be balanced by a resisting opposite couple created by torsional spring stress distributed over the cross-section of the spring which represented by (T=P * R) .

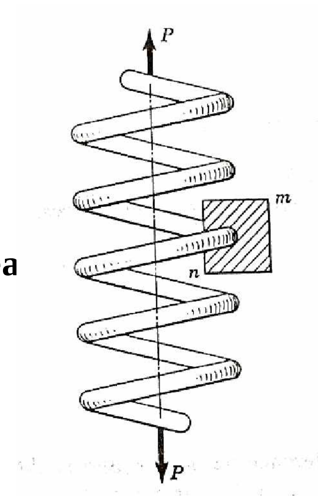


Fig.a

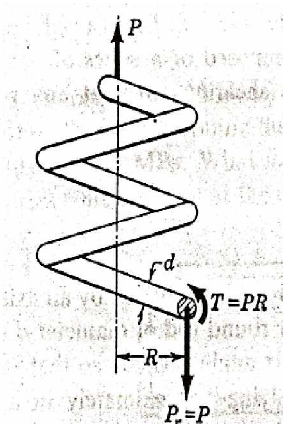


Fig.b

In figure (c) , there are two types of shearing stress are produced :

1- Direct shearing stresses (τ_1) uniformly distributed over spring section and creating the resisting load (P_1) that pass through the centre of section .

2- Variable torsional shearing stress (τ_2) caused by twisting couple (T=P * R) .

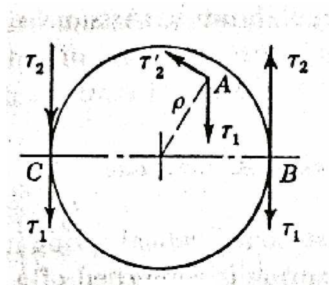


Fig.c

The torsional stresses (τ_2) vary in magnitude with their radial distance from the centre and are directed perpendicular to the radius as at point (A) .

The resultant shearing stress is the vector sum of the direct and torsional shear stress . At point (B) the stresses are oppositely directed and the resultant stress is the difference between (τ_1) and (τ_2) .

At point (C) inside fiber , the two stresses are collinear and in the same sense , there sum produces the maximum stress in the section .

Maximum direct shear stress $(\tau = \frac{P}{A})$

Maximum torsional shear stress $(\tau = \frac{T * r}{J})$

Maximum total shearing stress $(\tau = \tau_1 + \tau_2)$

$$\therefore \tau_{t\max.} = \frac{P}{\frac{\pi}{4} d^2} + \frac{P * R * (\frac{d}{2})}{\frac{\pi}{32} d^4}$$

$$\therefore \tau_{t\max.} = \frac{4P}{\pi d^2} + \frac{16 P * R}{\pi d^3}$$

$$\therefore \tau_{t\max.} = \frac{16 P * R}{\pi d^3}(1 + \frac{d}{4R}) \quad \text{------------(1)}$$

Equation (1) contains an error because the torsion formula derived for use with straight bar was applied here to a curved bar , this error is of significance in heavy springs . The importance of this error depend upon how greatly the inner and outer elements , differ in ordinary evidently this difference depends on how sharply curved the spring wire , i.e. upon the ratio of wire diameter (d) to mean radius (R).

A.M.Wahl has developed the following formula that takes account of the initial curvature of the spring wire :-

$$\tau = \frac{16 P * R}{\pi * d^3}\left[\frac{4m-1}{4m-4} + \frac{0.615}{m}\right] \quad \text{------------(2)}$$

where :- $(m = \frac{2R}{d})$ **The ratio of the mean diameter of the spring to the diameter of the spring wire**

$$\tau_{\max.} = \frac{16 * P * R}{\pi * d^3}\left[1 + \frac{0.5}{m}\right] \quad \text{------------(3)}$$

Spring are usually made of special steel and bronze in which the allowable shearing stresses rang from (200 to 800MPa) .

<u>Spring Deflection :-</u> Its defined as the elongation or contraction in spring and is determined from the following equation :-

$$\delta = \frac{64 * P * R * N}{G * d^{4}} \quad \text{------------(4)}$$

where :-

N--------- number of turns (coils) of radius (R) .

Equation (4) derived as follows :-

From figure(d) as angle (dθ) small , the length of arc (AD) is equal to (AB*dθ), and it may be considered as straight line perpendicular to (AB) , from the similarity triangles (ADE) and (BAC) we can obtain :

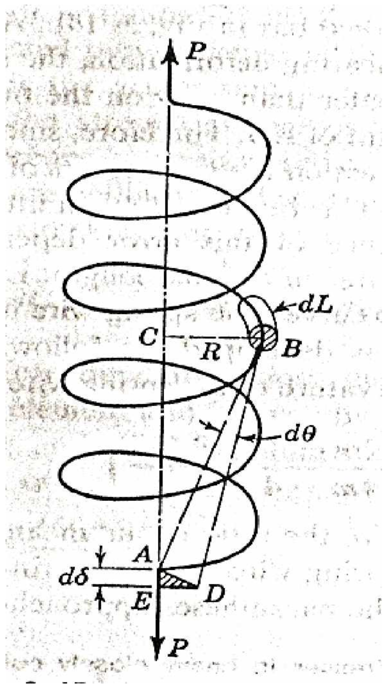


Fig.d

$$\frac{AE}{AD} = \frac{BC}{AB}$$

or $$\frac{d\delta}{AB * d\theta} = \frac{R}{AB}$$

where :- ($d\delta = R * d\theta$)

as $$\theta = \frac{T * L}{J * G} \Rightarrow d\theta = \frac{T * dL}{J * G}$$

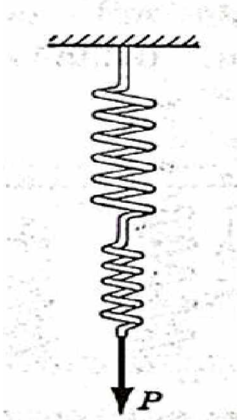
$$\therefore \frac{d\delta}{R} = \frac{(P * R)\,dL}{JG}$$

$$\therefore \delta = \frac{P * R^{2} L}{JG}$$

By integrating to give the total elongation contributed by all elements of spring by replacing the length of spring wire (L) by (2πRN) and replacing (J) by ($\frac{\pi}{32} d^{4}$)

$$\therefore \delta = \frac{64 * P * R^{3} * N}{G * d^{4}}$$

Ex:-5- **A load (*P*) is supported by two steel spring arranged in series as shown in figure . The upper spring has (20 turns) of (20mm) diameter wire on mean diameter of (150mm) . The lower spring consists of (15 turns) of (10mm) diameter wire on mean diameter of (130mm) . Determine the maximum shearing stress in each spring if the total deflection is (80mm) and (G_s=83GPa).**



Sol:-

$$\delta = \frac{64 * P * R^3 * N}{G * d^4}$$

$$0.08 = \frac{64 * P}{83 * 10^9}\left[\frac{(0.075)^3 (20)}{(0.02)^4} + \frac{(0.065)^3 * 15}{(0.01)^4}\right]$$

$$\therefore \ P = 233 \ N$$

For upper spring $(m = \frac{2R}{d} = \frac{2*0.075}{0.02} = 7.5)$

$$\tau_{max.} = \frac{16 P * R}{\pi * d^3}\left[\frac{4m - 1}{4m - 4} + \frac{0.615}{m}\right]$$

$$\tau_{max.} = \frac{16 * 233 * 0.075}{\pi * (0.02)^3}\left[\frac{30 - 1}{30 - 4} + \frac{0.615}{7.5}\right]$$

$$\therefore \tau_{max.} = 12.7 MPa$$

For lower spring $(m = \frac{2R}{d} = \frac{2*0.065}{0.01} = 13)$

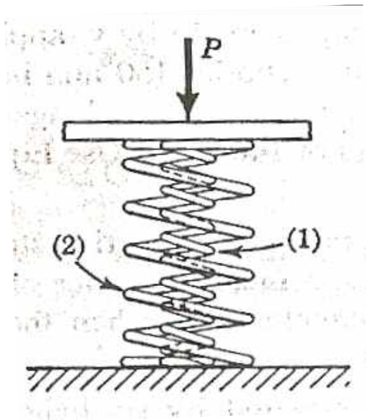
$$\tau_{max.} = \frac{16 P * R}{\pi * d^3}\left[\frac{4m - 1}{4m - 4} + \frac{0.615}{m}\right]$$

$$\tau_{max.} = \frac{16 * 233 * 0.065}{\pi * (0.01)^3}\left[\frac{52 - 1}{52 - 4} + \frac{0.615}{13}\right]$$

$$\therefore \tau_{max.} = 81 . MPa$$

Ex:-6- A load (*P*) is supported by two concentric steel spring arranged as shown in figure . The inner spring consist of (30 turns) of (20mm) wire diameter on mean diameter of (150mm) , the outer spring has (20 turns) of (30mm) wire diameter on mean diameter of (200mm) . Compute the maximum load that will not exceed a shearing stress of (140 MPa) in either spring . G_s=83 GPa .

Sol:-



$$\delta_1 = \delta_2 \quad , \quad \delta = \frac{64P * R^3 * N}{G * d^4}$$

$$\frac{64P_1 * (0.075)^3 * 30}{G * (0.02)^4} = \frac{64P_2 * (0.1)^3 * 20}{G * (0.03)^4}$$

P_1=0.312 P_2

For inner spring $(m = \frac{2R}{d} = \frac{2 * 0.075}{0.02} = 7.5)$

For outer spring $(m = \frac{2R}{d} = \frac{2 * 0.1}{0.03} = 6.67)$

$$\tau_{max.} = \frac{16\, P * R}{\pi * d^3}\left[\frac{4m - 1}{4m - 4} + \frac{0.615}{m}\right]$$

For (P_1) :-

$$140 * 10^6 = \frac{16 * P_1 * 0.075}{\pi * (0.02)^3}\left[\frac{30 - 1}{30 - 4} + \frac{0.615}{7.5}\right]$$

P_1=2.45 kN and P_2= 7.85kN $\Rightarrow$ P= P_1+ P_2=10.3 kN

For (P_2) :-

$$140 * 10^6 = \frac{16 * P_2 * 0.1}{\pi * (0.03)^3}\left[\frac{26.7 - 1}{26.8 - 4} + \frac{0.615}{6.67}\right]$$

P_2=6.06 kN and P_1= 1.9 kN $\Rightarrow$ P= P_1+ P_2=7.96 kN

$\therefore$ **P= 7.96 kN**