Chapter Eight

Combined Stresses

The three basic types of loading are :

1- Axial Loading : This loading lead to axial stress (Tension or Compression)

$$\sigma_{ax.} = \pm \frac{P}{A}$$

2- Torsional Loading : This loading lead to torsional stress

$$\tau_{T} = \pm \frac{T * r}{J}$$

3- Bending (Flexural) Loading : This loading lead to bending stress (flexural stress) .

$$\sigma_{be.} = \pm \frac{M * y}{I}$$

In some times the bending stress due to shear stress called (shear due to bending) .

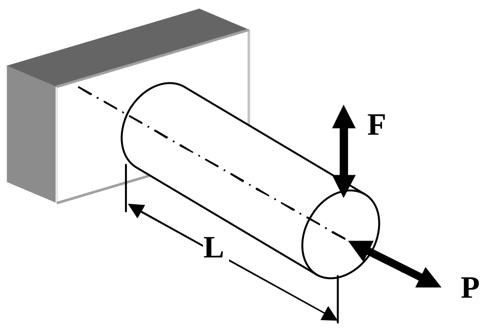
$$\tau = \pm \frac{V}{I * b} A' * y'$$

The possible combination of loading may be shown in four forms :

1) **Combined axial and bending loading** :

$$\sigma_{ax} = \pm \frac{P}{A}$$

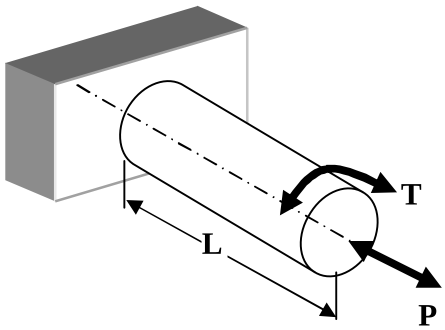
$$\sigma_{be.} = \pm \frac{M * y}{I} \quad , \quad M = \pm F * L$$



and may subjected to shear due to bending

$$\tau = \pm \frac{V}{I * b} A' * y'$$

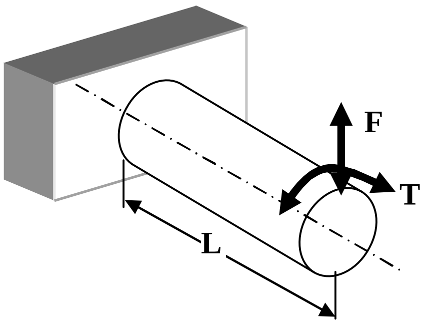
2) **Combined axial and torsional loading** :



$$\sigma_{ax} = \pm \frac{P}{A}$$

$$\tau_T = \pm \frac{T * r}{J}$$

3) **Combined bending and torsional loading:**



$$\sigma_{be} = \pm \frac{M * y}{I} \;, \; M = \pm F * L$$

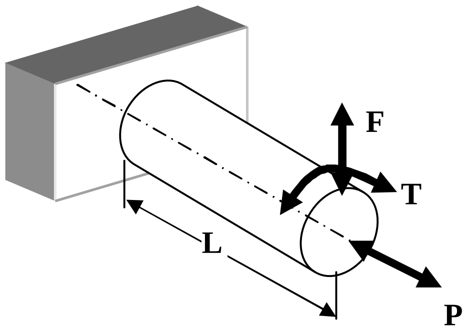
$$\tau_T = \pm \frac{T * r}{J}$$

and may subjected to shear due to bending

$$\tau = \pm \frac{V}{I * b} A' * y'$$

4) **Combined axial , bending and torsional loading:**

$$\sigma_{ax} = \pm \frac{P}{A}$$



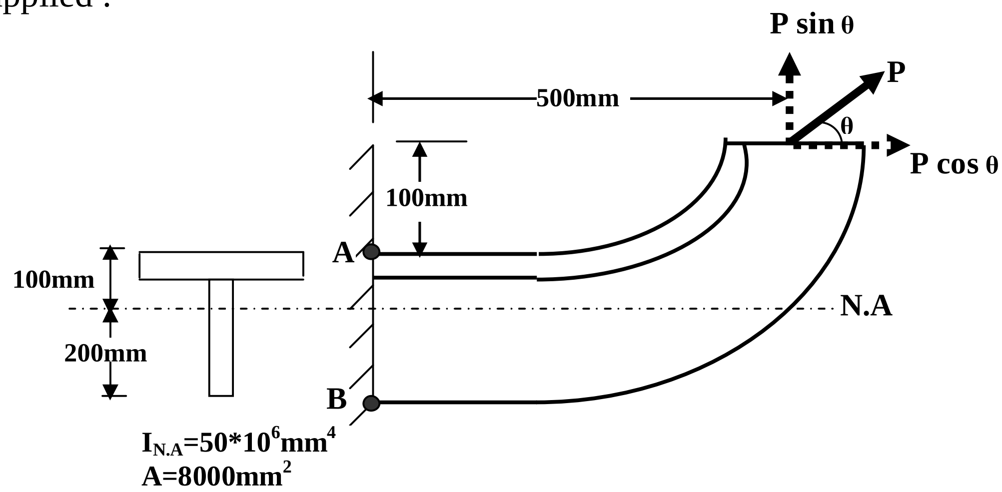
$$\sigma_{be} = \pm \frac{M * y}{I} \;, \; M = \pm F * L$$

$$\tau_T = \pm \frac{T * r}{J}$$

and may subjected to shear due to bending

$$\tau = \pm \frac{V}{I * b} A' * y'$$

Ex: For bracket shown in figure find the stresses at point (A) and (B) when load (P=100kN) applied .



Sol :

M_{AB}= (100*sin 37^o)*0.5 - (100*cos 37^o)*0.2

M_{AB}=14 kN.m

At point (A) :

$$\sigma_A = \frac{P}{A} - \frac{M * y}{I}$$

$$\sigma_A = \frac{100 * 10^3 \cos 37}{8000 * 10^{-6}} - \frac{14 * 10^3 * 0.1}{50 * 10^{-6}}$$

$$\sigma_A = -18 MPa$$

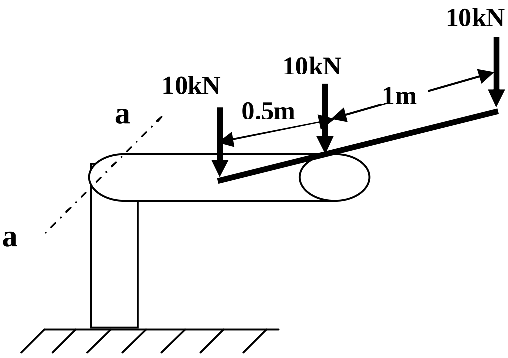
At point (B) :

$$\sigma_B = \frac{P}{A} + \frac{M * y}{I}$$

$$\sigma_B = \frac{100 * 10^3 \cos 37}{8000 * 10^{-6}} + \frac{14 * 10^3 * 0.2}{50 * 10^{-6}}$$

$$\sigma_B = 66 MPa$$

Ex :- Find the maximum shear stress at section (a-a) for shaft of (100mm) diameter and loaded as shown in figure .



Sol :-

$$\tau_d = \frac{V}{I*b} A'*y'$$

$$\tau_d = \frac{3*10*10^3}{\frac{\pi}{64}(0.1)^4*0.1}\left[\frac{\pi}{8}*(0.1)^2*\left(\frac{2}{3\pi}d\right)\right]$$

$$\tau_d = 5.1 MPa$$

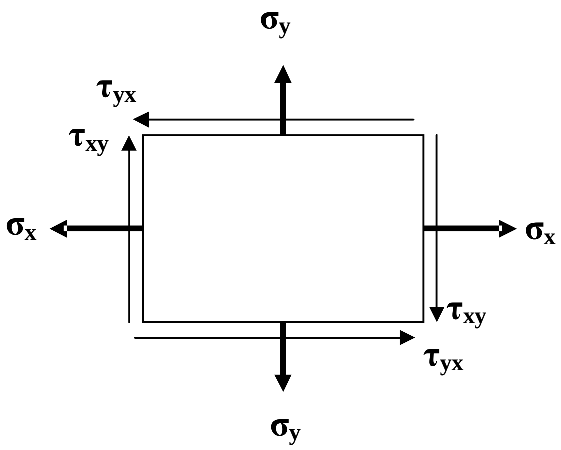
$$\tau_T = \frac{T*r}{J} \quad , \quad T= 10*1 - 10*0.5 = 5kN.m$$

$$\tau_T = \frac{5*10^3*0.05}{\frac{\pi}{32}(0.1)^4} = 25.46\, MPa$$

$$\tau_{max.} = \tau_d + \tau_T = 5.1 + 25.46 = 30.56\, MPa$$

Stress at Point :

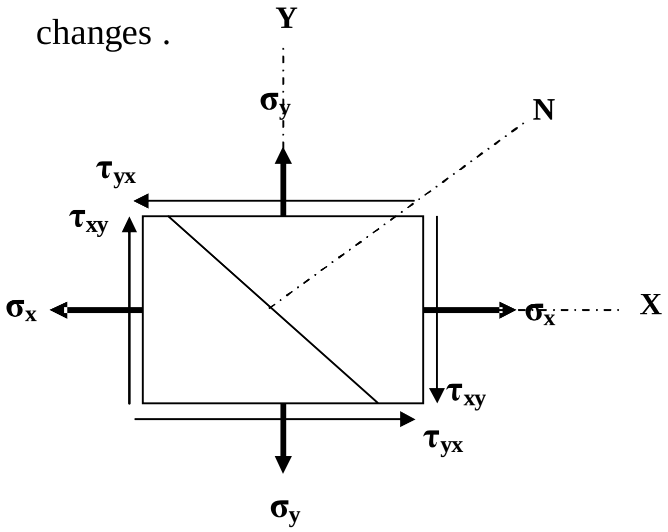
The uniform stress distributed over differential area and the stress acting on element is usually represent by front view (2D view) and the stresses acting on the element are known as stress component .



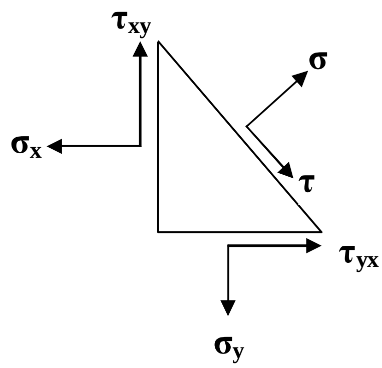
1) Variation of Stress at Point : (Analytical Solution)

The stress acting at a point is represented by the stresses acting on the faces of the element enclosing the point .

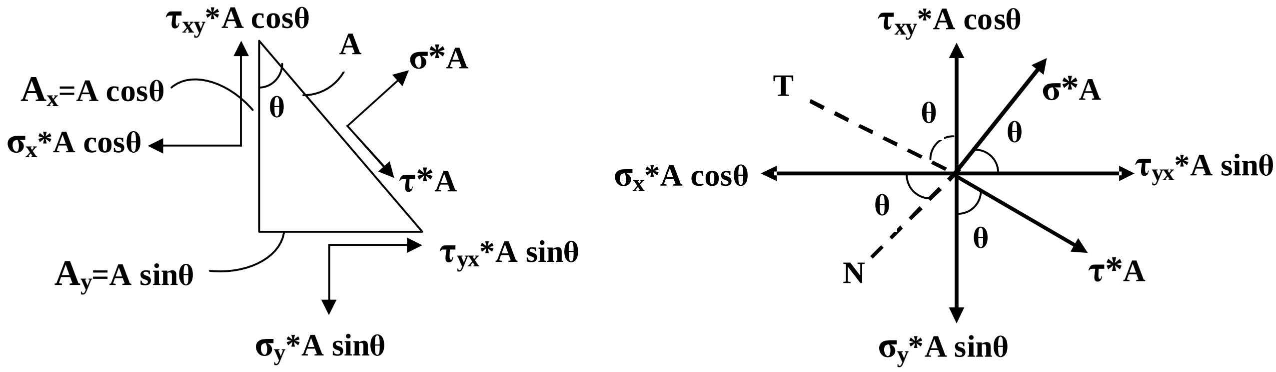
These stresses change with inclination of planes passing thought that point , i.e. the stresses on the faces of the element vary as the angular position of the element changes .



a) Original state of stresses



b) Stresses acting on wedge



c) Free-body diagram of forces on wedge

d) Point diagram of forces

For equilibrium Forces on element :-

$$\sum T = 0 \Rightarrow \tau A = (\sigma_x A\cos\theta)\sin\theta - (\sigma_y A\sin\theta)\cos\theta + (\tau_{xy} A\cos\theta)\cos\theta - (\tau_{yx} A\sin\theta)\sin\theta \quad \text{--------(1)}$$

Can canceled term (A) from both sides and (τ_{xy}) equal to (τ_{yx}) and also

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad , \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}, \text{ and } \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

and

$$\sum N = 0 \Rightarrow \sigma A = (\sigma_x A\cos\theta)\cos\theta + (\sigma_y A\sin\theta)\sin\theta - (\tau_{xy} A\cos\theta)\sin\theta - (\tau_{yx} A\sin\theta)\cos\theta \quad \text{--------(2)}$$

Equation (1) can be reduced to :

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{--------(3)}$$

and equation (2) can be reduced to :

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{--------(4)}$$

The planes defining maximum or minimum normal stresses are found by differentiating equ.(4) with respect to (θ) and setting the derivative equal to zero .

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

similarly, the planes of maximum shearing stress are defined by :

$$\tan 2\theta = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Hence the planes on which maximum and minimum normal stress occur on plane of zero shearing stress . The maximum and minimum normal stress are called principal stresses .

In other words the plane of maximum shearing stress at (45^o) with planes of principal stress .

$$\therefore \sigma_{\substack{\max.\\ \min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max.} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

2) Mohr s Circle : (Graphical Solution)

This circle was constructed by German engineer (Otto Mohr) in 1882 .
If this circle plotted to scale , the results can be obtained graphically .

$$\sigma - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \textbf{--------(1)}$$

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad \textbf{--------(2)}$$

By squiring both these equations , adding the result and simplifying we obtain :

$$(\sigma - \frac{\sigma_x + \sigma_y}{2})^2 + \tau^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau^2_{xy} \qquad \textbf{--------(3)}$$

Where : (σ_x , σ_y , and τ_{xy}) are constant and (σ and τ) are various .

Let : $\frac{\sigma_x + \sigma_y}{2} = C(constant)$

The right hand side of equation (3)is equal to (R)

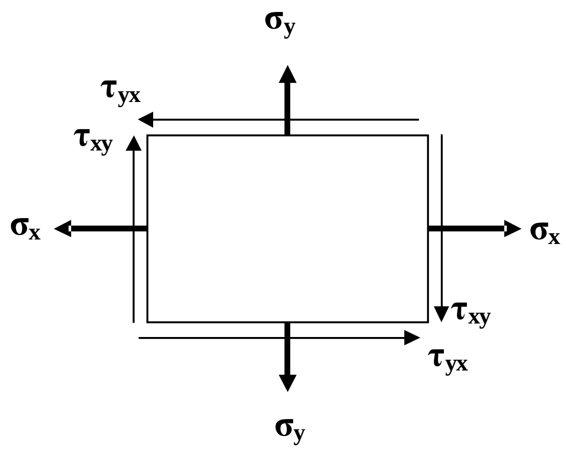
$$\therefore (\frac{\sigma_x - \sigma_y}{2})^2 + \tau^2_{xy} = R^2$$

$$\therefore (\sigma - C)^2 + \tau^2 = R^2$$

$$\therefore (X - C)^2 + Y^2 = R^2$$

$$\therefore R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau^2_{xy}}$$ as: (R) --------- radius of Mohr s circle

$$C = \frac{\sigma_x + \sigma_y}{2}$$ as: (C) Centre is offset rightward distance .


σy
τyx
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σx
σx
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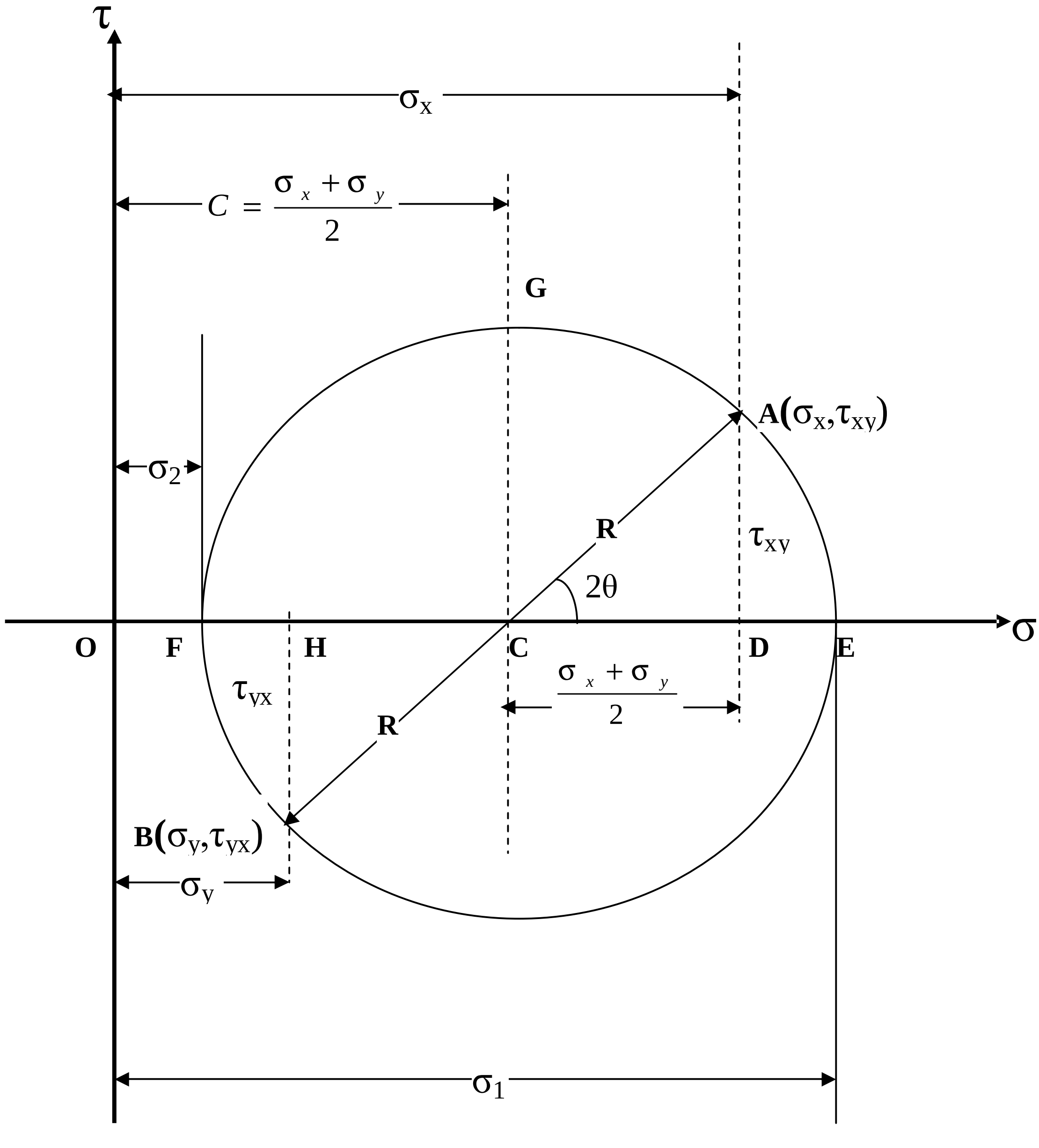
σy


τ
σx
C = (σx + σy)/2
G
A(σx,τxy)
σ2
R
τxy
2θ
σ
O
F
H
C
D
E
(σx + σy)/2
τyx
R
B(σy,τyx)
σy
σ1

Rules for Applying Mohr's Circle to Combined Stress :

1- On rectangular (σ-τ) axis , plot points having the coordinates as point (A) is (σ_x , τ_{xy}) and point (B) is (σ_y , $\tau_{y,x}$) , represents the normal and shearing stresses acting on the (X) and (Y) faces of an element for which the stresses are known , as the tension is (+ve) and compression is (-ve) and shearing stress as (+ve) when its rotate about the centre of element as clockwise and (-ve) when rotate as counterclockwise .

2- Join the points just plotted by straight line , that is the diameter of circle whose centre on (σ- axis) .

3-As different planes are passed through the selected point in stressed body , the normal and shearing stress component on these planes are represented by the coordinates of points whose position shifts around the circumference of Mohr's circle .

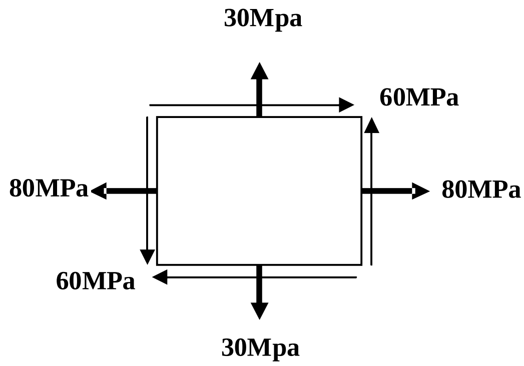
4- The radius of the circle to any point on its circumference represents the axis directed normal to the plane whose stress components are given by the coordinates of that point .

5- The angle between the radii to selected points on Mohr's circle is twice the angle between the normal to the actual planes represented by these point or to twice the space angularity between the planes so represented . The rotational sense of this angle corresponding to the rotational sense of the actual angle between the normal to the planes .

Ex :- For the state of stress shown in figure , determine the principal stresses and maximum shearing stress , and show all results on complete sketch of differential element.

Sol:

1) Analytical solution :



$$\therefore \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$$

$$\sigma_1 = \frac{80+30}{2} + \sqrt{(\frac{80-30}{2})^2 + (60)^2} = 120\,MPa$$

$$\sigma_2 = \frac{80+30}{2} - \sqrt{(\frac{80-30}{2})^2 + (60)^2} = -10\,MPa$$

$$\tau_{max.} = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2} = R$$

$$\tau_{max.} = \sqrt{(\frac{80-30}{2})^2 + (60)^2} = 65\,MPa$$

$$\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{-2*60}{80-30} = -2.4 \Rightarrow 2\theta = -67^o.4 \Rightarrow \theta = -33^o.7$$

2) Graphical solution :

A (80 , -60) , B (30 , 60)

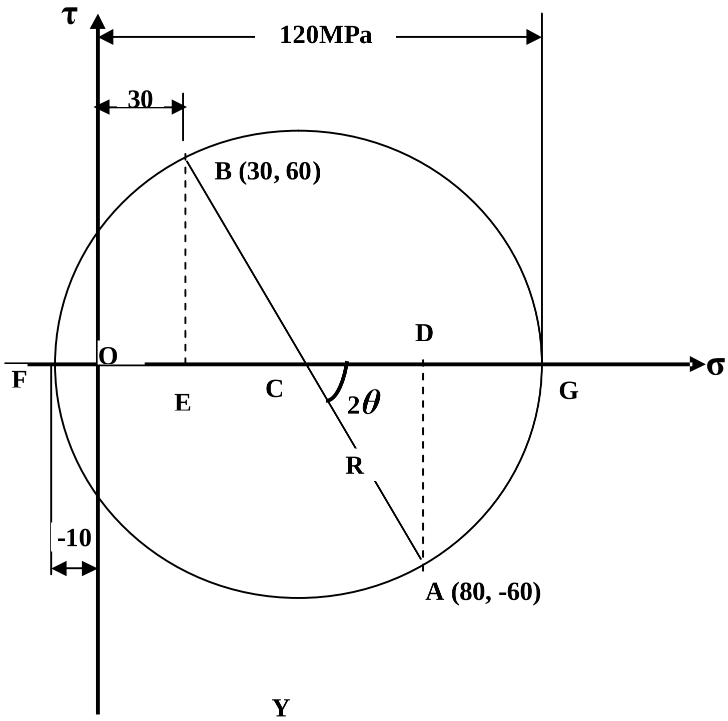
$$OC = \frac{\sigma_x + \sigma_y}{2} = \frac{80+30}{2} = 55 MPa$$

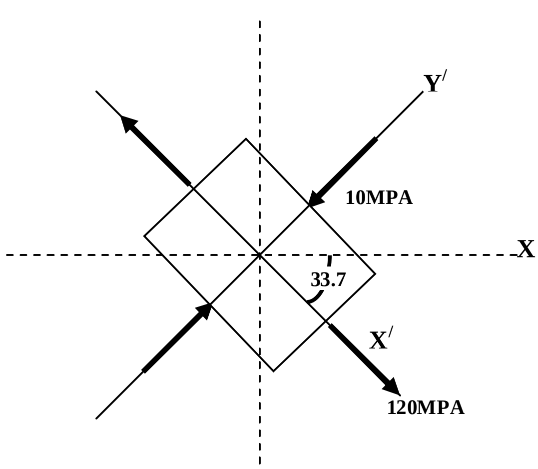
$$\tau_{max.} = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2} = R$$

$$\tau_{max.} = \sqrt{(\frac{80-30}{2})^2 + (60)^2} = 65\,MPa$$

$$\sigma_{\max.} = \sigma_1 = OG = OC + R = 55 + 65 = 120\,MPa$$
$$\sigma_{\min.} = \sigma_2 = OF = OC - R = 55 - 65 = -10\,MPa$$

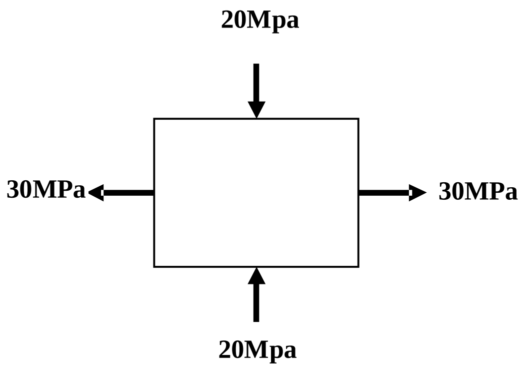
$$2\theta = Sin^{-1}\frac{\tau_{xy}}{R} = Sin^{-1}\frac{-60}{65} \Rightarrow \theta = -33^o.7$$





Ex:- For a state of principal stresses shown , find the normal and shearing stresses when angle between the radii is (31^o) .

Sol:-



$$R = \frac{|\sigma_1| + |\sigma_2|}{2} = \frac{30+20}{2} = 25MPa = \tau_{max.}$$

$$\sigma_1 = R + OC \Rightarrow OC = \sigma_1 - R = 30 - 25 = 5MPa$$

$$Sin\ 2\theta = \frac{\tau_{xy}}{R} \Rightarrow Sin\ 62^o = \frac{\tau_{xy}}{25}$$

$$\therefore \tau_{xy} = 22\ MPa$$

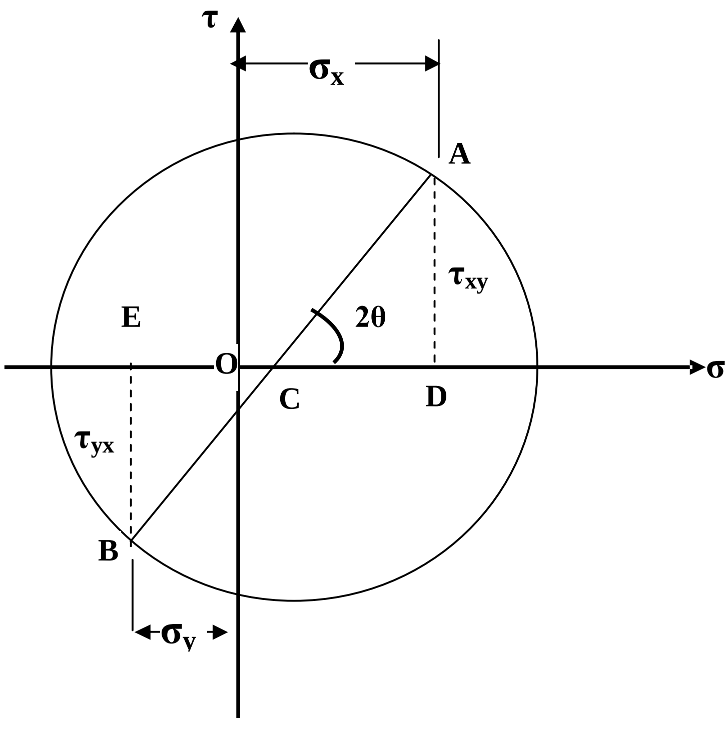
$$\sigma_x = OC + CD = OC + RCos\ 62^o$$

$$\sigma_x = 5 + 25\ Cos\ 62^o = 16.74\ MPa$$

$$\sigma_y = -RCos\ 2\theta + 5 = -6.74\ MPa$$

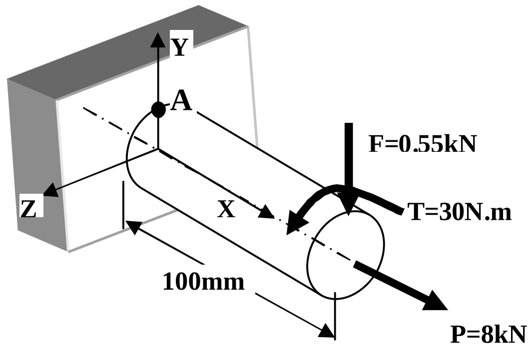
$$\tau_{xy} = \tau_{yx} = 22\ MPa$$

$\therefore$ A (16.74 , 22) , B (-6.74 , 22)



Ex:- A cantilever shaft (d=20mm) is subjected to tensile load of (P=8 kN) and bending force of (F=0.55 kN) and torque of (30 N.m) . Find the principal stresses and maximum shearing stress and show all results on complete sketch .

Sol:-



$$\sigma_{ax.} = \frac{P}{A} = \frac{8*10^3}{\frac{\pi}{4}(0.02)^2} = 25.5MPa$$

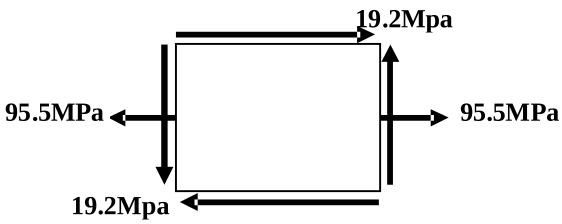
$$M_Z = F * L = 0.55 * 10^3 * 0.1 = 55\ N.m$$

$$\sigma_{bend} = \frac{M * y}{I} = \frac{55 * \frac{0.02}{2}}{\frac{\pi}{64}(0.02)^4} = 70\ MPa$$

$$\sigma_x = \sigma_{ax.} + \sigma_{bend.} = 25.5 + 70 = 95.5\,MPa$$

$$\tau_T = \frac{T * r}{J} = \frac{-30 * 0.01}{\frac{\pi}{32}(0.02)^4} = -19.24\ MPa$$

State of stress can plotted as shown



A(95.5 , -19.2) , B (0 , 19.2)

$$OC = \frac{\sigma_x + \sigma_y}{2} = \frac{95.5 + 0}{2} = 47.75MPa$$

$$\tau_{max.} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = R$$

$$\tau_{max.} = \sqrt{\left(\frac{95.5 - 0}{2}\right)^2 + (19.24)^2} = 51.5\,MPa$$

$$\sigma_{max.} = \sigma_1 = OC + R = 47.75 + 51.5 = 99.25\,MPa$$

$$\sigma_{min.} = \sigma_2 = OC - R = 47.75 - 51.5 = -3.75\,MPa$$

$$2\theta = Sin^{-1}\frac{\tau_{xy}}{R} = Sin^{-1}\frac{-19.24}{51.5} \Rightarrow \theta = -11^{o}$$