Chapter Five

Stresses in Beams

In this chapter we will derive the relations between the bending moment and the flexure (bending) stress it causes and between the vertical shear force and the shearing stress .

In deriving these relations , the following assumptions are used :

1- Plane sections of the beam , originally plane and remain plane .

2- The material of beam is homogenous and obeys Hook's law .

3- The modulus of elasticity in tension and compression are equal .

4- The beam is initially straight and of constant cross-section .

5- The plane of loading must contain a principal axis of the beam cross-section and the loads must be perpendicular to the longitudinal axis of the beam .

Flexure(Bending) Stress (σ_b) : **The stresses which are caused by bending moment.**

Flexure Formula : **The relation between flexure stress and bending moment .**

Neutral Surface : **The plain containing fibers of beam which remain unchanged in length and hence carry no stress .**

Neutral Axis (N.A) : **The line of intersection between the transverse cross-section.**

The beam in figure (1) shows two section (ab) and (cd) that separate by distance (*dx*) .

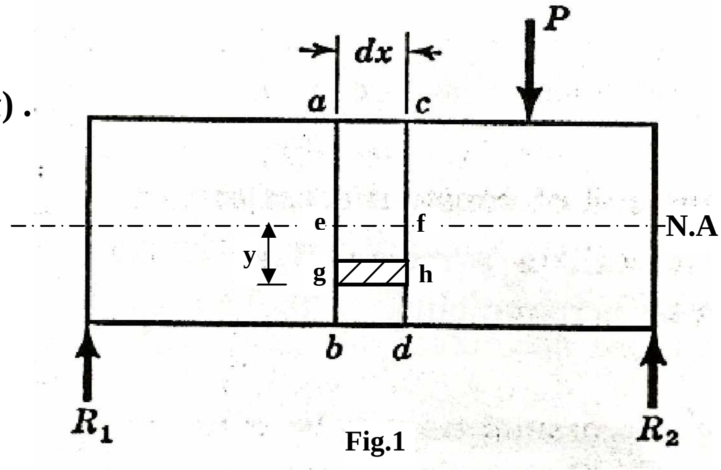


Fig.1

Figure (2) show the deflected shape of the beam .

$$\delta = hk = yd\theta$$

$$Strain = \frac{\delta}{L} = \frac{yd\theta}{ef}$$

$$ef = \rho d\theta$$

$$\therefore \varepsilon = \frac{yd\theta}{\rho d\theta} \Rightarrow \varepsilon = \frac{y}{\rho}$$

$$\sigma = E * \varepsilon$$ **(Hook's law)**

$$\sigma = E\,\frac{y}{\rho}$$

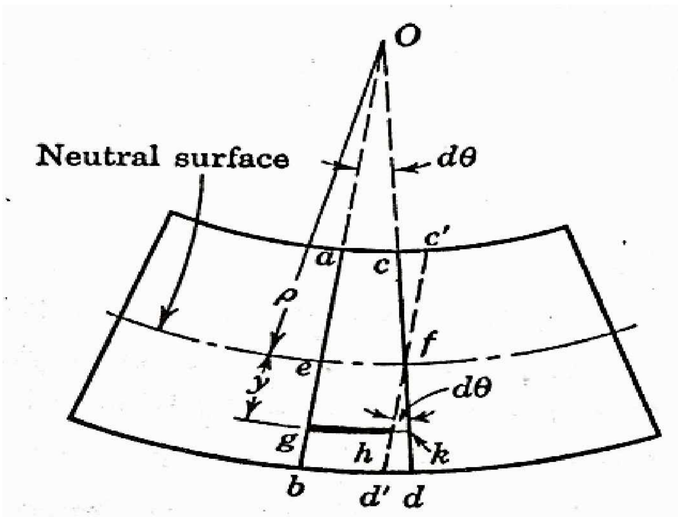


Fig.2

$$\sum F_x = 0$$

$$\therefore \int \sigma_x * dA = 0$$

$$n\ \sigma = \frac{E}{\rho} y$$

$$\therefore \frac{E}{\rho} \int y * dA = 0$$

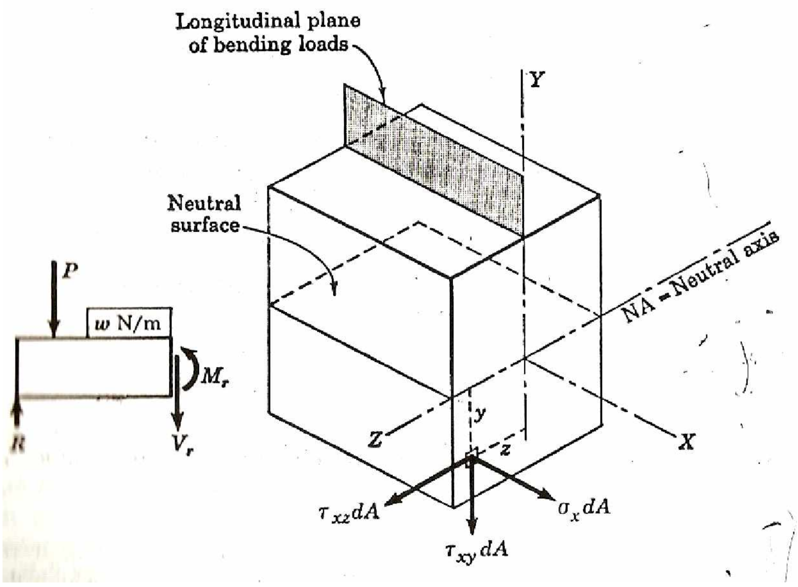


Fig. 3

Since the (*ydA*) is the moment of the differential area (*dA*) about the neutral axis , the integral ($\int ydA$) is the total moment of area

hence : $$\frac{E}{\rho} A y' = 0$$

only (y') can be equal zero , i.e. the neutral axis must contain the centroid of the cross-sectional area .

$\sum F_y = 0$ **that leads to the shear stress formula (**$V_r = \int \tau_{xy} * dA$**)**

$\sum F_z = 0$ **That leads to the formula of shear (**$\tau_{xz} * dA = 0$ **)**

$$\sum M_y = 0 \Rightarrow \int z(\sigma_x * dA) = 0$$

$$n\ \sigma = \frac{E * y}{\rho} \Rightarrow \frac{E}{\rho} \int z * y * dA = 0$$

The integral ($\int z * y * dA$ **) is the product of inertia (*P*P_{zy}) , which is zero only if (*y*) or (*z*) is an axis of symmetry or principal axis .**

$$\sum M_z = 0 , M = M_r$$

$$M = \int y(\sigma_x * dA)$$

$$\therefore\ M = \frac{E}{\rho}\int y^2 * dA$$

Since ($y^2 * dA$) is defined as moment of inertia (I)

$$\therefore M = \frac{E * I}{\rho}$$

$$\therefore \frac{E}{\rho} = \frac{M}{I} = \frac{\sigma}{y}$$

$$\therefore \sigma = \frac{M * y}{I}$$

$$\sigma_{max} = \frac{M*C}{I}$$, **where *C*------maximum value of (*y*)**

if ($\frac{I}{C}$) is called the section modulus (*S*)

$$\therefore \sigma_{max.} = \frac{M}{S}$$

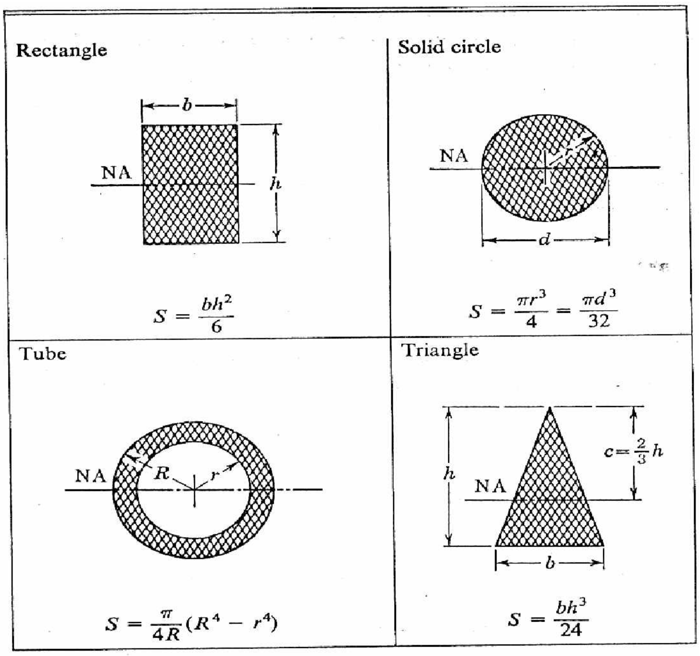
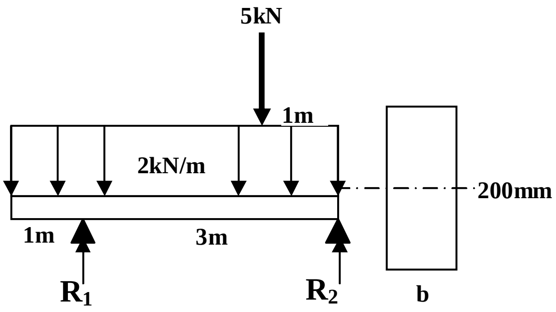


Fig. 4 The section modulus

Ex: -1- Determine the minimum width (b) of the beam shown if bending (flexural) stress is not to exceed (10MPa) .



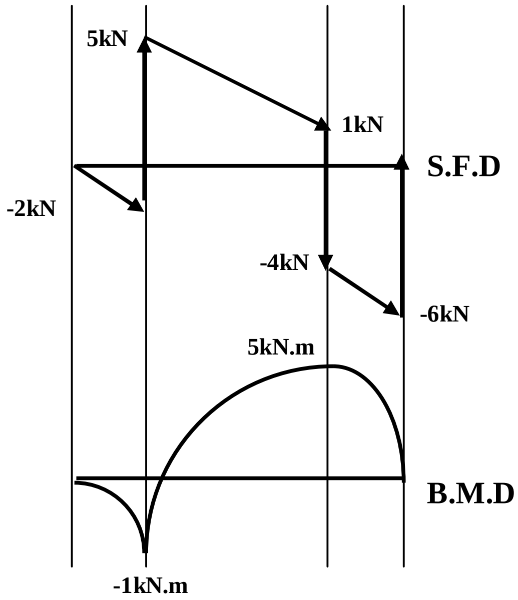
Sol:-

$$\sum M_{R_2} = 0$$

$$3R_1 = 2*4*2 + 5*1 \therefore R_1 = 7kN$$

$$\sum F_y = 0 \Rightarrow R_1 + R_2 = 2*4+5$$

$$R_2 = 6kN$$



$$0 \le x \le 1m$$

$$V = -2x \Rightarrow x = 0 \Rightarrow V = 0$$

$$x = 1 \Rightarrow V = -2kN$$

$$M = -2x*\frac{x}{2} = -x^2 \Rightarrow x = 0 \Rightarrow M = 0$$

$$x = 1 \Rightarrow M = -1kN.m$$

$$0 \le x \le 3m$$

$$V = -2x + 7 \Rightarrow x = 1 \Rightarrow V = 5kN$$

$$x = 3 \Rightarrow V = 1kN$$

$$M = -2x*\frac{x}{2} + 7(X-1) = -x^2 + 7(x-1)$$

$$\therefore x = 1 \Rightarrow M = -1kN.m$$

$$x = 3 \Rightarrow M = 5kN.m$$

$$0 \le x \le 4m$$

$$V = -2x + 7 - 5 = -2x + 2 \Rightarrow x = 3 \Rightarrow V = -4kN$$

$$x = 4 \Rightarrow V = 6kN$$

$$M = -2x*\frac{x}{2} + 7(X-1) - 5(x-3) = -x^2 + 7(x-1) - 5(x-3)$$

$$\therefore x = 3 \Rightarrow M = 5kN.m$$

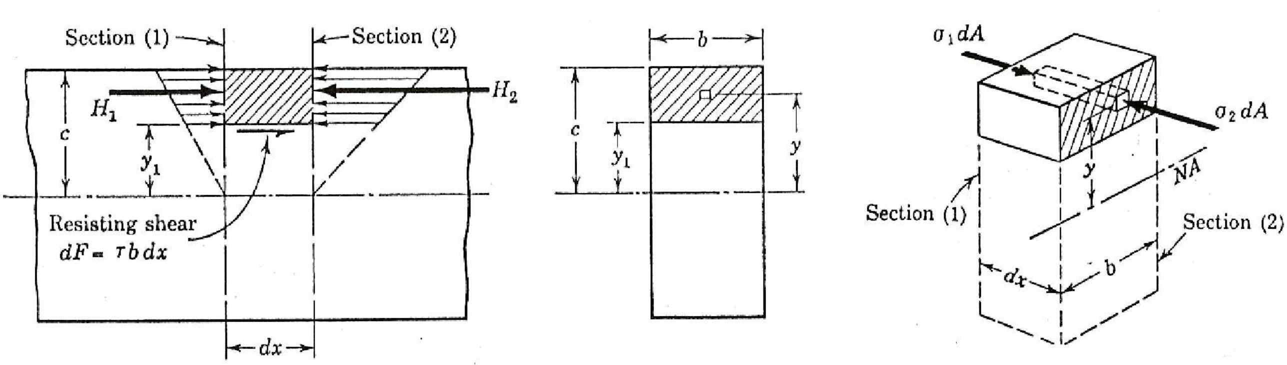
$$x = 4 \Rightarrow M = 0$$

$$\sigma_{max.} = \frac{M_{max.} * y}{I} , I = \frac{bh^3}{12}$$

$$\frac{b * (0.2)^3}{12} = \frac{5 * 10^3 * 0.1}{10 * 10^6}$$

$$\therefore b = 75 \ mm$$

<u>Derivation of formula for horizontal shear stress :</u>



Assume B.M at section (2) to be larger than that at section (1) , thus causing larger bending (flexural) stress at section (2) than at section(1) .

The horizontal resultant thrust (H_2) caused by compressive forces on section (2) will be greater than horizontal resultant thrust (H_1) on section (1) . The difference between (H_2) and (H_1) can be balanced only by the resisting shear force (*dF*) acting on the bottom face of free body .

$$\sum F_x = 0 \Rightarrow dF = H_2 - H_1 = \int_{y_1}^{c} \sigma_2 * dA - \int_{y_1}^{c} \sigma_1 * dA$$

but $\sigma = \dfrac{M*y}{I}$

$$\therefore dF = \frac{M_2}{I} \int_{y_1}^{c} y * dA - \frac{M_1}{I} \int_{y_1}^{c} y * dA$$

$$\therefore dF = \frac{M_2 - M_1}{I}\int_{y_1}^{c} y * dA \quad \text{but} \quad dF = \tau * b * dx$$

$$\therefore dF = \frac{M_2 - M_1}{I}\int_{y_1}^{c} y * dA = \tau * b * dx$$

$$\therefore \tau = \frac{dM}{I * b * dx}\int_{y_1}^{c} y * dA$$

$$\text{but} \quad \frac{dM}{dx} = V \quad \text{(shear force)}$$

$$\therefore \tau = \frac{V}{I * b}\int_{y_1}^{c} y * dA$$

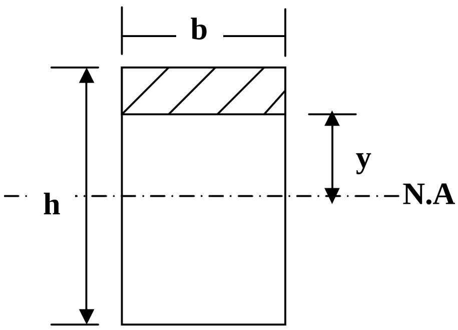
$$\therefore \tau = \frac{V}{I * b} * A' * y' = \frac{V}{I * b} * Q$$

where :

A' ---------**area over N.A**

y' -------- **distance from (** A' **) center to (N.A)**

For rectangular section:-



$$\tau = \frac{V}{I * b} A' * y'$$

$$\tau = \frac{V}{I * b}\left[b(\frac{h}{2} - y)\right]\left[y + \frac{1}{2}(\frac{h}{2} - y)\right]$$

$$\tau = \frac{V}{2I}(\frac{h^2}{4} - y^2)$$

Maximum shear stress at N.A

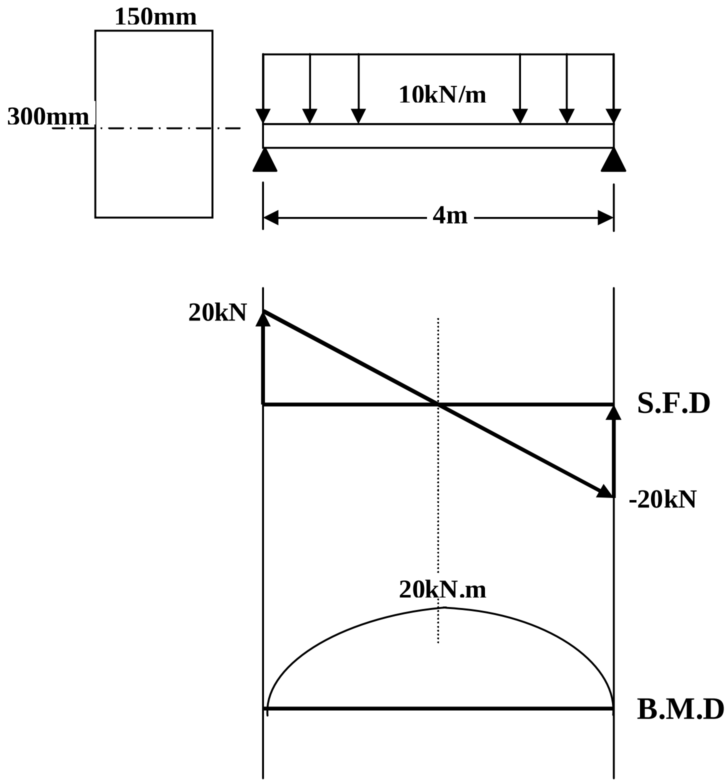
$$\tau = \frac{V}{I * b} A' * y' = \frac{V}{(\frac{b * h^3}{12})b} (\frac{b * h}{2}) * (\frac{h}{4})$$

$$\tau = \frac{3}{2} . \frac{V}{b * h} = \frac{3}{2} . \frac{V}{A}$$

$\therefore$ The maximum shearing stress in rectangular section is (50%) greater than average shear stress .

Ex:- 2- A rectangular beam of (150*300mm) loaded by uniformly distributed load of (10kN/m) , find the maximum bending(flexural) and shearing stresses .

Sol :-



$$\sum F_y = 0 \Rightarrow R_1 + R_2 = 10 * 4$$

$$\therefore R_1 = R_2 = 20kN$$

$$0 \le x \le 4$$

$$\therefore V = 20 - 10x$$

at $x = 0 \Rightarrow V = 20kN$

at $x = 4 \Rightarrow V = -20kN$

$$M = 20x - 10x * \frac{x}{2}$$

$$x = 0 \Rightarrow M = 0$$

$$x = 4 \Rightarrow M = 0$$

$$\frac{dM}{dx} = 20 - 10x = 0 \Rightarrow x = 2m$$

$$M_{x=2m} = 20kN.m$$

$$\therefore \sigma_{max.} = \frac{M * y}{I} = \frac{20 * 10^3 * 0.15}{\frac{0.15 * (0.3)^3}{12}} = 8.9MPa$$

$$\tau = \frac{V}{I * b} * A' * y' = \frac{20 * 10^3 (0.15 * 0.15)}{\frac{0.15 * (0.3)^3}{12} * 0.15} * 0.075$$

$$\therefore \tau = 0.67 \ MPa$$