Chapter Six

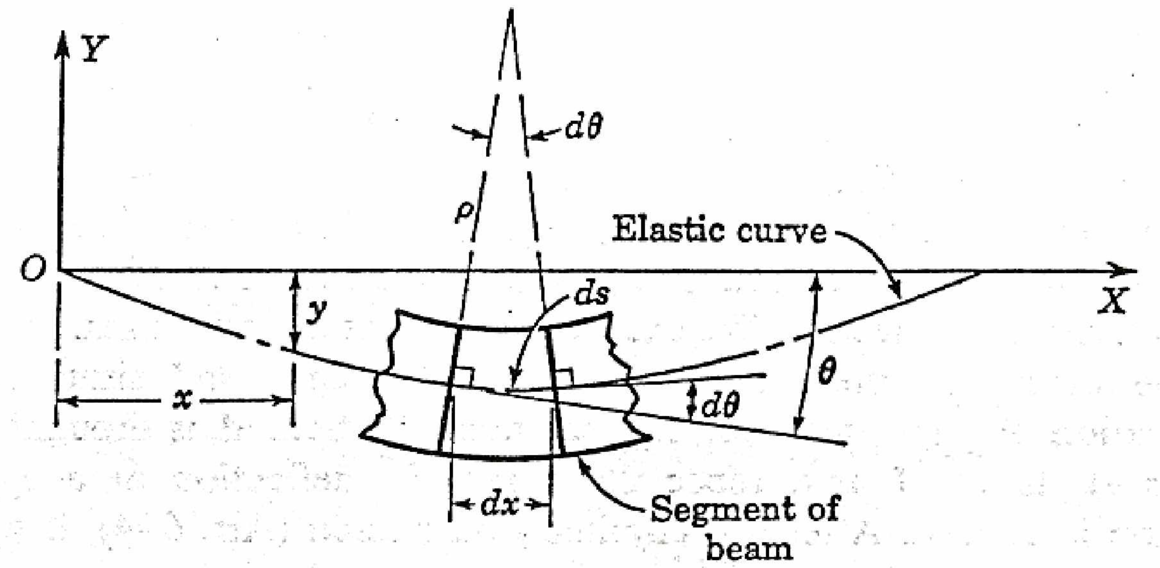
Beam Deflection

1) Double Integration Method :

The edge view of the neutral surface of deflected beam is called the *elastic curve* of the beam , it is shown greatly exagerated in the figure below. This article shows how to determine the equation of the curve , i.e. how to determine the vertical displacement (y) of any point in terms of its (x) coordinate .

The deflections are assumed to be so small that there is no appreciable difference between the original length of the beam and the projection of its deflected length . The elastic curve is very flat and its slop at any point is very small .

The value of the slop $(\tan\theta = \frac{dy}{dx})$ **may therefore with only small error be set equal to (** θ **) .**



$\tan\theta = \frac{dy}{dx}$ **as (θ) is small $\Rightarrow$ tan $\theta = \theta$**

$$\theta = \frac{dy}{dx} \text{ --------------(1)}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \text{ --------------(2)}$$

$$ds = \rho * d\theta \text{ --------------(3)}$$

ρ -------- radius of curvature

as elastic curve is very flat

$$\therefore ds \approx dx$$

$$\frac{1}{\rho} = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \text{ ----------(4)}$$

$$\therefore \frac{1}{\rho} = \frac{d^2y}{dx^2} \text{ --------------(5)}$$

but $\frac{1}{\rho} = \frac{M}{E*I}$

$$\therefore EI\frac{d^2y}{dx^2} = M \text{ ------------(6)}$$

This equation called differential equation

(E*I)---------- Bending (Flexural) rigidity of beam and is constant ($N.m^2$)

$$\therefore \frac{d^2y}{dx^2} = \frac{M}{E*I}$$

$$M = E*I\frac{d^2y}{dx^2}$$

$$\therefore E*I\frac{dy}{dx} = \int M * dx + C_1$$

as $\left(\dfrac{dy}{dx}\right)$ **is slop of elastic curve**

$$\therefore\ E * I * y = \iint M * dx + C_1 * x + C_2$$

as y**-------- beam deflection (m or mm)**

C_1, C_2 -------- The constant of integration

<u>Note :-</u> At the supports the value of deflection for beam equal to zero (y = 0)

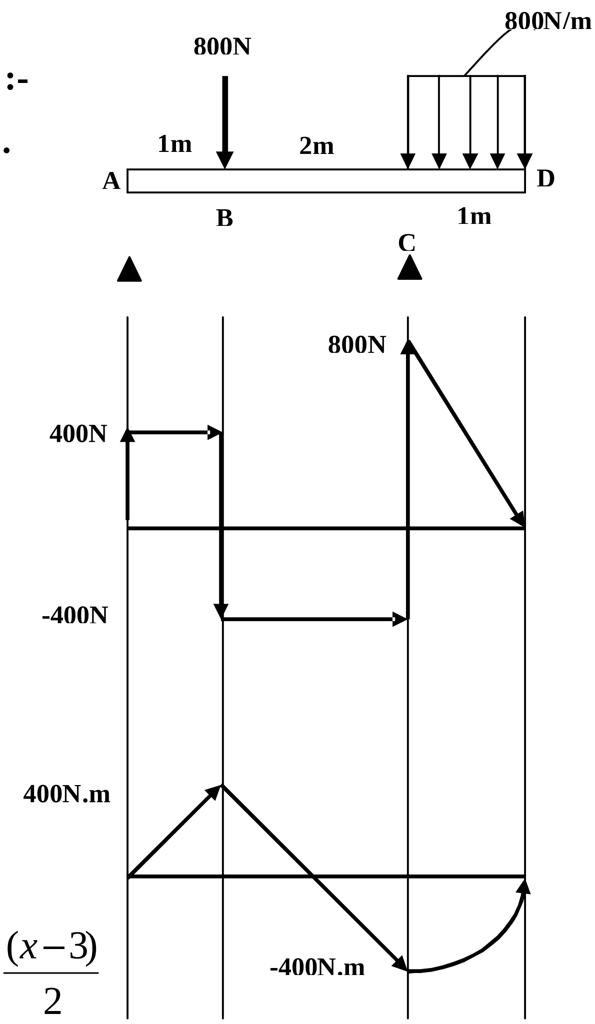
<u>Ex</u>: (1)

For the beam loaded as shown in figure , find :-

1- The slop of elastic curve over right support .

2- The deflection between the support .

Sol:-



$$\sum F_y = 0 \Rightarrow R_A + R_C = 800 + 800 = 1600N$$

$$\sum M_A = 0 \Rightarrow 3R_C = 800*1 + 800*1*3.5$$

$$\therefore R_C = 1200\,N \Rightarrow R_A = 400\,N$$

$$0 \le x \le 1 \Rightarrow M = 400x$$

$$\therefore x = 0 \Rightarrow M = 0$$

$$x = 1 \Rightarrow M = 400N.m$$

$$0 \le x \le 3 \Rightarrow M = 400x - 800(x-1)$$

$$\therefore x = 1 \Rightarrow M = 400N.m$$

$$x = 3 \Rightarrow M = -400N.m$$

$$0 \le x \le 4$$

$$\therefore M = 400x - 800(x-1) + 1200(x-3) - 800(x-3)\frac{(x-3)}{2}$$

$$\therefore x = 3 \Rightarrow M = 400N.m$$

$$x = 4 \Rightarrow M = 0$$

$$E*I\frac{d^2y}{dx^2}=M=400x-800(x-1)+1200(x-3)-400(x-3)^2$$

$$E*I\frac{dy}{dx}=200x^2-400(x-1)^2+600(x-3)^2-\frac{400}{3}(x-3)^3+C_1$$

$$E*I*y=\frac{200}{3}x^3-\frac{400}{3}(x-1)^3+200(x-3)^3-\frac{400}{12}(x-3)^4+C_1x+C_2$$

$x=0\Rightarrow y=0$

$x=3\Rightarrow y=0$ **deflection equal zero at the supports**

$C_2=0$

$0=1800-1066.66+3C_1\Rightarrow C_1=-244.45$

$$\therefore E*I*y=\frac{200}{3}x^3-\frac{400}{3}(x-1)^3+200(x-3)^3-\frac{400}{12}(x-3)^4-244.45x$$

slop at right support (*x*=*3*)

$$E*I\frac{dy}{dx}=200(3)^2-400(3-1)^2+600(3-3)^2-\frac{400}{3}(3-3)^3-244.45*3$$

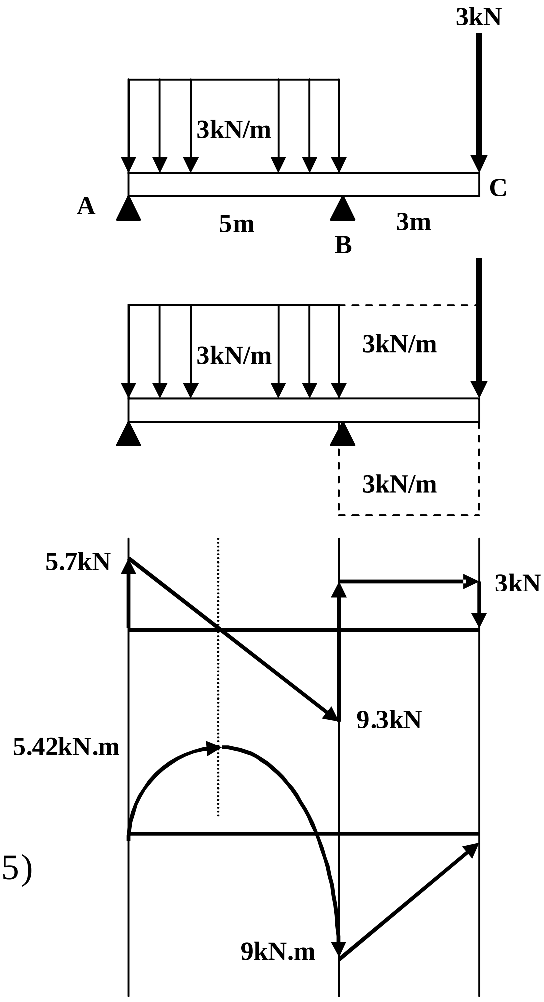
$$\therefore\frac{dy}{dx}=\frac{-44.45}{E*I}$$

Maximum deflection between the support at (x=1.5m)

$$E*I*y=\frac{200}{3}(1.5)^3-\frac{400}{3}(1.5-1)^3+\cancelto{\text{zero}}{200(1.5-3)^3}-\cancelto{\text{zero}}{\frac{400}{12}(1.5-3)^4}-244.45*1.5$$

$$\therefore y=\frac{-158.34}{E*I}(m)$$

Ex:-(2) For the beam shown in figure find the beam deflection at the right end .



Sol:-

$$\sum F_y = 0 \Rightarrow 3 + 3*5 = R_A + R_B$$

$$\sum M_A = 0 \Rightarrow 3*5*2.5 + 3*8 - R_B * 5 = 0$$

$$R_B = 12.3kN \Rightarrow R_A = 5.7kN$$

$$0 \le x \le 5 \Rightarrow V = 5.7 - 3x$$

$$x = 0 \Rightarrow V = 5.7kN$$

$$x = 5 \Rightarrow V = -9.3kN$$

$$M = 5.7x - 3x * \frac{x}{2} = 5.7x - \frac{3}{2}x^2$$

$$x = 0 \Rightarrow M = 0$$

$$x = 5 \Rightarrow M = -9kN.m$$

$$\frac{dM}{dx} = 5.7 - 3x = 0 \Rightarrow x = 1.9m$$

$$\therefore M_{x=1.9m} = 5.42kN.m$$

$$0 \le x \le 8 \Rightarrow V = 5.7 - 3x + 3(x-5) + 12.3$$

$$x = 5 \Rightarrow V = 3kNx = 8 \Rightarrow V = 3kN$$

$$M = 5.7x - 3x * \frac{x}{2} + 3(x-5)\frac{(x-5)}{2} + 12.3(x-5)$$

$$M = 5.7x - \frac{3}{2}x^2 + \frac{3}{2}(x-5)^2 + 12.3(x-5)$$

$$x = 5 \Rightarrow M = -9kN.m, x = 8 \Rightarrow M = 0$$

$$E*I\frac{d^2y}{dx^2} = 5.7x - \frac{3}{2}x^2 + \frac{3}{2}(x-5)^2 + 12.3(x-5)$$

$$E*I\frac{dy}{dx} = \frac{5.7}{2}x^2 - \frac{3}{6}x^3 + \frac{3}{6}(x-5)^3 + \frac{12.3}{2}(x-5)^2 + C_1$$

$$E*I*y = \frac{5.7}{6}x^3 - \frac{3}{24}x^4 + \frac{3}{24}(x-5)^4 + \frac{12.3}{6}(x-5)^3 + C_1x + C_2$$

$$x = 0 \Rightarrow y = 0 \Rightarrow C_2 = 0,$$

$$x = 5 \Rightarrow y = 0 \Rightarrow C_1 = -40.6$$

$$E*I*y = \frac{5.7}{6}x^3 - \frac{3}{24}x^4 + \frac{3}{24}(x-5)^4 + \frac{12.3}{6}(x-5)^3 - 40.6x$$

$$E*I*y_{x=8m} = -305.2kN.m^3$$

2) Theorem of Area Moment Method :

A useful and simple method of determining slopes and deflection in beams involves the area of moment diagram and also the moment of the area by *area moment method* .

$$\frac{1}{\rho} = \frac{M}{E*I}$$

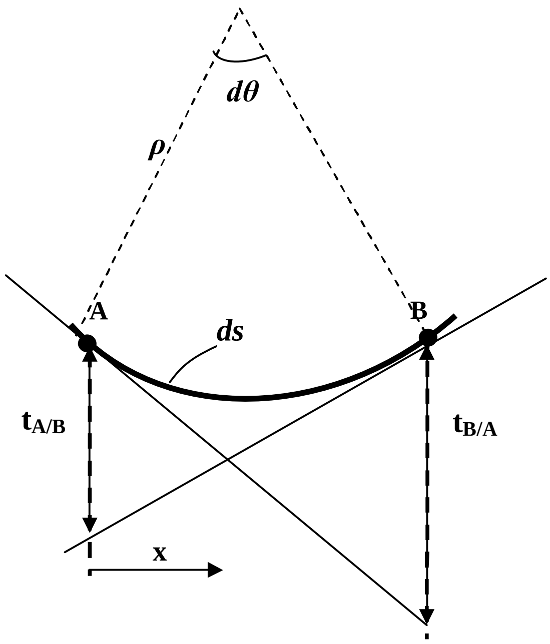
but $ds = \rho\, d\theta$

$$\therefore \frac{1}{\rho} = \frac{M}{E*I} = \frac{d\theta}{ds} \Rightarrow d\theta = \frac{M}{E*I} ds$$

but $ds \approx dx$ **(Flat curve)**

$$\therefore d\theta = \frac{M}{E*I} dx$$

$$\therefore \theta = \frac{1}{E*I} \int_{x_1}^{x_2} M * dx$$



$$dt = x * d\theta$$

$$t_{B/A} = \int dt = \int x * d\theta$$

$$t_{B/A} = \frac{1}{E*I} \int_{x_1}^{x_2} x\,(Mdx)$$

$t_{B/A}$---------- Deviation of (B) from a tangent drawn at (A) or the tangential deviation of (B) with respect to (A) .

but $\int M * dx$ **is the summation of area .**

$$\therefore \theta_{B/A} = \frac{1}{E*I} (area)_{BA}$$

$$\therefore t_{B/A} = \frac{1}{E*I} (area)_{BA} * \bar{x}_B$$

Moment Diagram By Parts :

The construction of moment diagram by parts depends on two basic principles :

1) The resultant bending moment at any section caused by any load system is the algebraic sum of the bending moment at that section caused by each load acting separately .

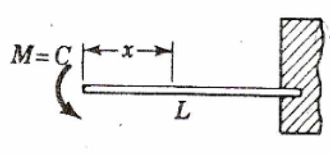
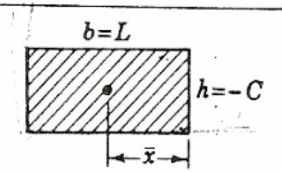
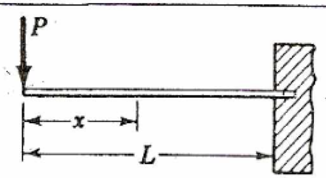
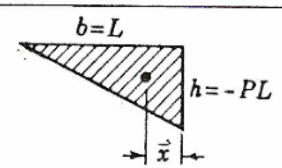
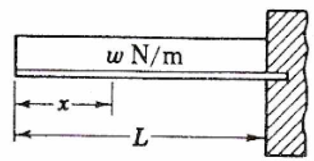
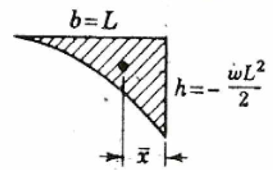
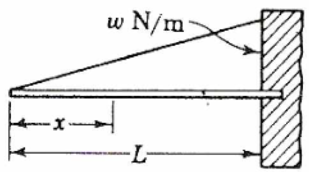
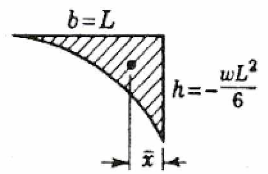
$$M = \sum M_L = \sum M_R$$

$\sum M_L , \sum M_R$ **---------- Sum of the moment caused by all the forces to the left and right section respectively .**

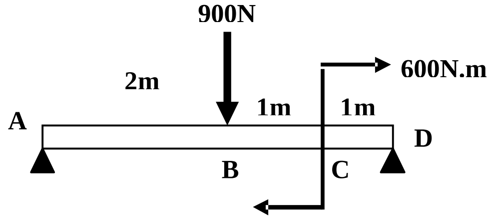
2) The moment effect of any single specified loading is always some variation of the general equation .

$$\boxed{\begin{aligned} area &= \frac{1}{n+1} b * h \\ \bar{x} &= \frac{1}{n+2} * b \end{aligned}}$$

TABLE Cantilever Loadings

TYPE OF LOADING	CANTILEVER BEAM	MOMENT EQUATION (moment at any section x)	DEGREE OF MOMENT EQUATION	MOMENT DIAGRAM	AREA	$\bar{x}$
Couple		$M = -C$	Zero (i.e., $M = -Cx^0$)		$\frac{1}{1}bh$	$\frac{1}{2}b$
Concentrated		$M = -Px$	1st		$\frac{1}{2}bh$	$\frac{1}{3}b$
Uniformly distributed		$M = -\frac{w}{2}x^2$	2nd		$\frac{1}{3}bh$	$\frac{1}{4}b$
Uniformly varying		$M = -\frac{w}{6L}x^3$	3rd		$\frac{1}{4}bh$	$\frac{1}{5}b$

Ex-1- Find the moment area of moment diagram for beam loaded as shown in figure .

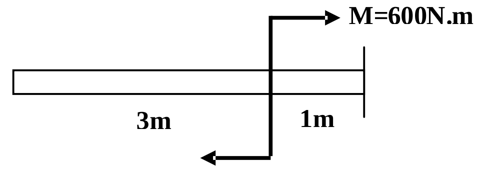


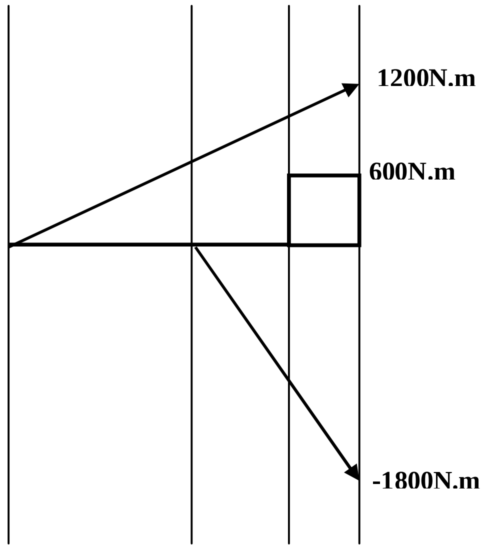
Sol:-

$$\sum F_y = 0 \Rightarrow R_A + R_D = 900N$$

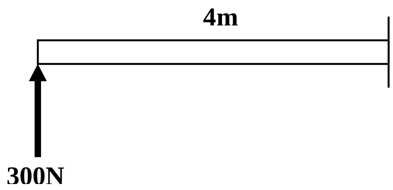
$$\sum M_D = 0 \Rightarrow 4R_A = 900*2-600$$

$$\therefore R_A = 300N \Rightarrow R_D = 600N$$

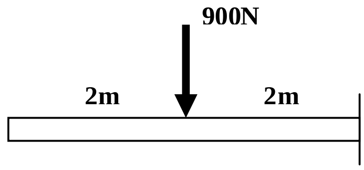




M=600N.m



M=300N*4m=1200N.m



M=-900N*2m=-1800N.m

$$E*I*t = (area)_{D/A} * \overline{x} = \sum area * \overline{x}$$

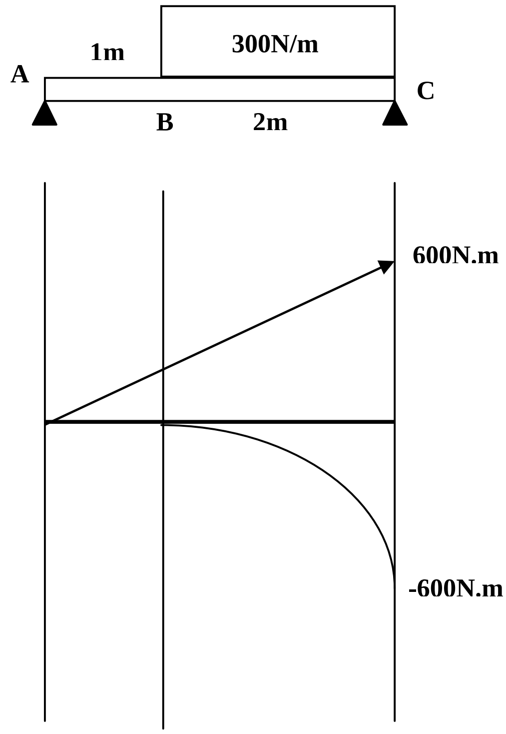
$$(area)_{D/A} * \overline{x} = [b*h*\frac{b}{2} + \frac{b}{2}h*\frac{b}{3} - \frac{b}{2}h*\frac{b}{3}]$$

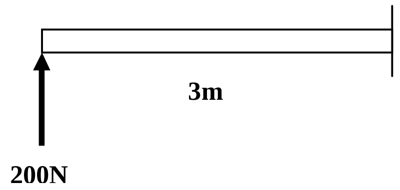
$$(area)_{D/A} * \overline{x} = [(600*1)*\frac{1}{2} + \frac{1}{2}4*1200*\frac{4}{3} - \frac{2*1800}{2}*(\frac{2}{3})]$$

$$(area)_{D/A} * \overline{x} = 2300\ N.m^3$$

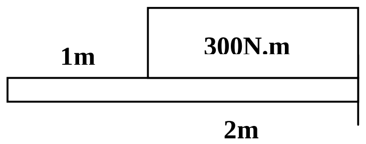
Ex-2- For the beam shown in figure , find the moment area about point (C) and the slope of elastic curve .

$$\sum F_y = 0 \Rightarrow R_A + R_C = 600N$$
$$\sum M_C = 0 \Rightarrow 3R_A = 300*2*1$$
$$\therefore R_A = 200N \Rightarrow R_C = 400N$$





M=200*3=600N.m



M=-300*2*1=-600N.m

$$E*I*t = (area)_{C/A} * \bar{x} = \sum area * \bar{x}$$

$$(area)_{C/A} * \bar{x} = [\frac{b}{2} * h * \frac{1}{3} b] + [\frac{1}{3} b * h * \frac{b}{4}]$$

$$(area)_{C/A} * \bar{x} = [(\frac{1}{2} * 3 * 600) * \frac{1}{3} * 3] + [(\frac{1}{3} 2 * (-600)) * \frac{1}{4} * 2]$$

$$(area)_{C/A} * \bar{x} = 700\ N.m^3$$

$$\theta = \frac{(area)_{C/A}}{E*I} = [\frac{1}{2} * 3 * 600 - \frac{1}{3} * 2 * 600]$$

$$\theta = \frac{500}{E*I} (rad)$$