Subject Name: Calculus I2

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Lecture No. 4

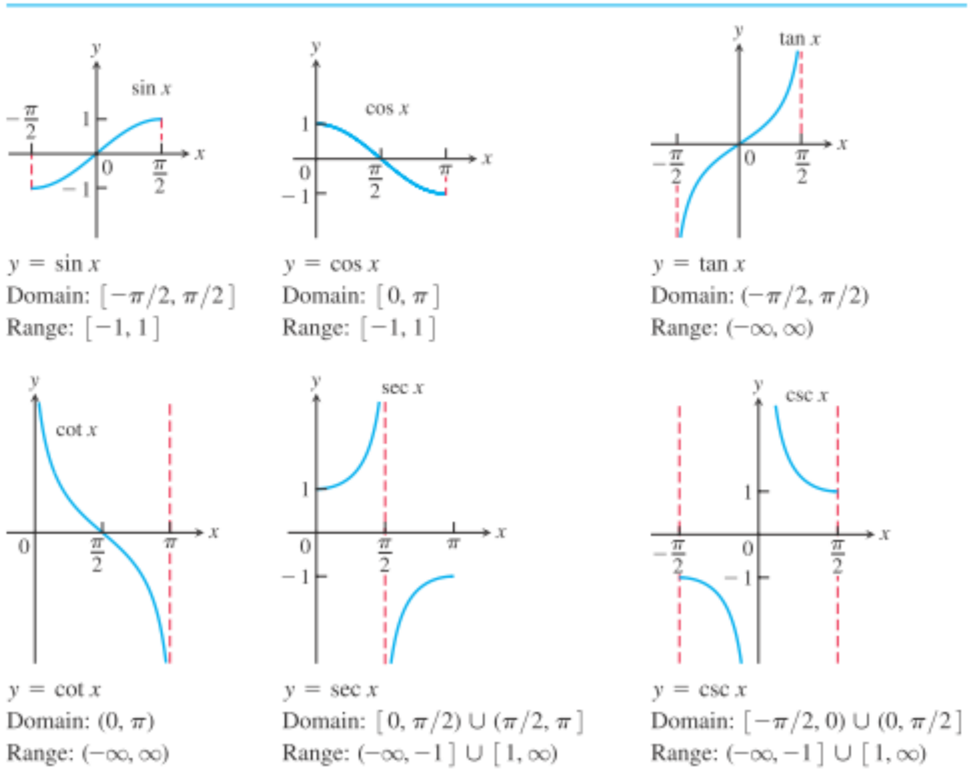
Lecture Title: Inverse Trigonometric Functions

Inverse Trigonometric Functions

Inverse trigonometric functions take place when we want to calculate angles from side measurements in triangles. However, their domains can be restricted to intervals on which they are **one-to-one functions**.

The six basic trigonometric functions are not one-to-one (since their values repeat periodically).

Domain restrictions that make the trigonometric functions one-to-one



Since these restricted functions are now one-to-one, they have inverses, which are denoted by:

$$y = \sin^{-1} x$$

$$y = \cos^{-1} x$$

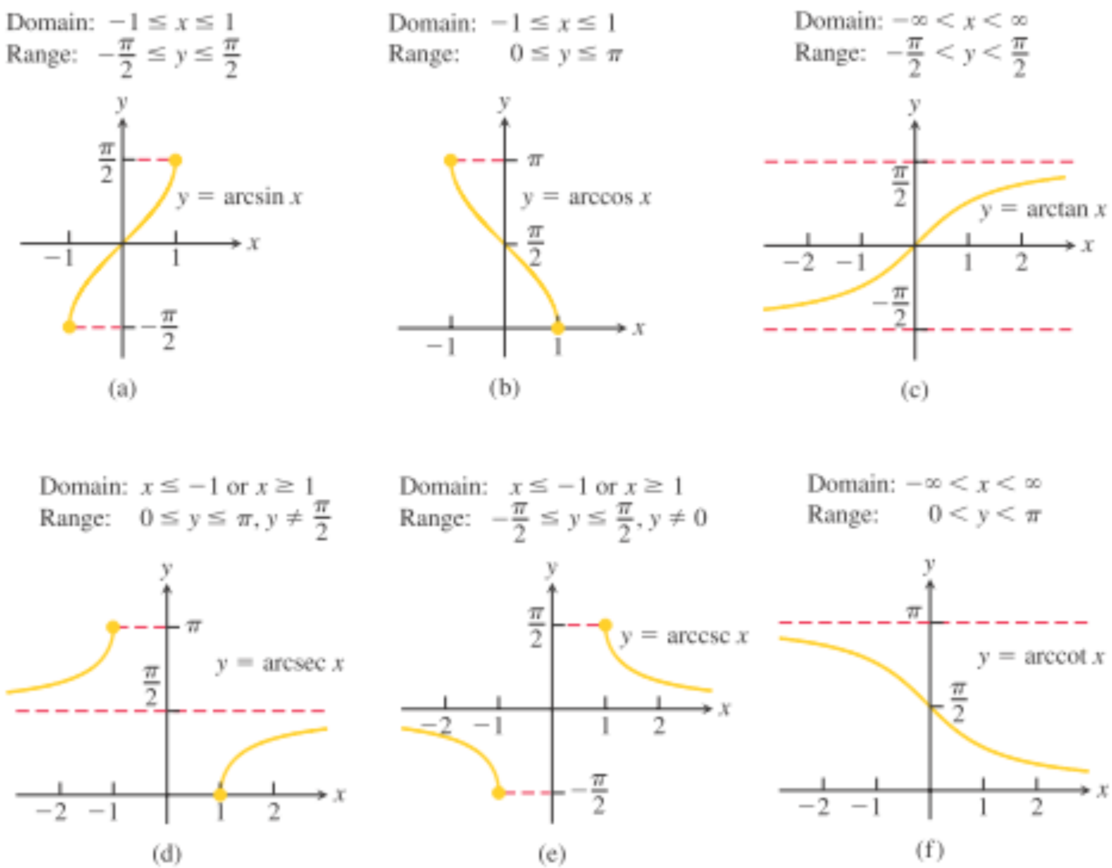
$$y = \tan^{-1} x$$

$$y = \cot^{-1} x$$

$$y = \sec^{-1} x$$

$$y = \csc^{-1} x$$

The Figures below show the graphs of all six functions.



The Arcsine and Arccosine Functions

The graph of $y = \arcsin x$ is symmetric about the origin .Therefore the arcsine is an odd function:

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

While the graph of $y = \cos^{-1} x$ has no such symmetry.

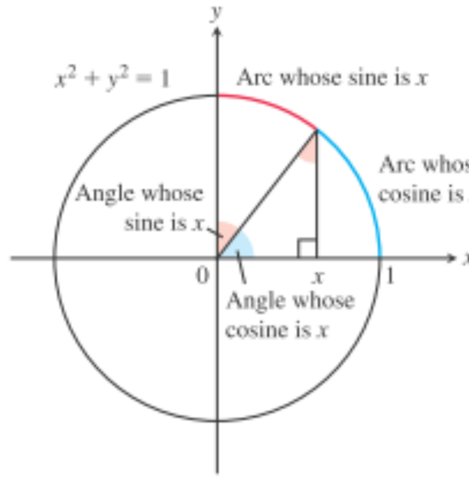
$$\sin^{-1}(x) \text{ is the number in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ for which } \sin y = x$$

$$\cos^{-1}(x) \text{ is the number in } [0, \pi] \text{ for which } \cos y = x$$

The 'Arc' in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is *x*, *y* is also the length of the arc on the unit circle that subtends an angle whose sine is *x*. So we call *y* "**the arc whose sine is x**."

$$s = \cos^{-1} x \quad \text{or} \quad s = \sin^{-1} x$$



Example 1: Evaluate a) arcsin$\left(\frac{\sqrt{3}}{2}\right)$,and b) arccos$\left(-\frac{1}{2}\right)$

Solution:

a) We see that:

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

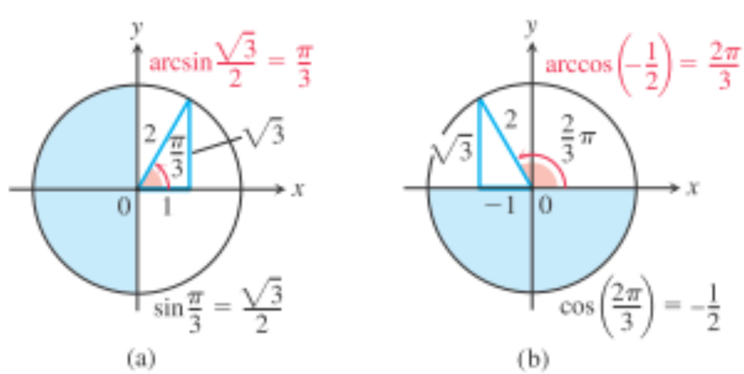
Because $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and the angle $\frac{\pi}{3}$ belongs to the range [-π/2, π/2] of the arcsine function.

b) We see that:

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Because $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and the angle $\frac{2\pi}{3}$ belongs to the range [0, π] of the arccos function.

Using the same procedure illustrated in **Example 1**, we can create the following table of common values for the arcsine and arccosine functions.

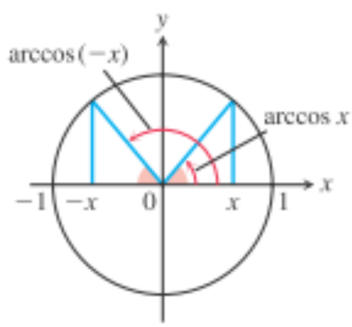


x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	π/3	π/6
$1/\sqrt{2}$	π/4	π/4
1/2	π/6	π/3
-1/2	-π/6	2π/3
$-1/\sqrt{2}$	-π/4	3π/4
$-\sqrt{3}/2$	- π/3	5 π/6

Identities Involving Arcsine and Arccosine

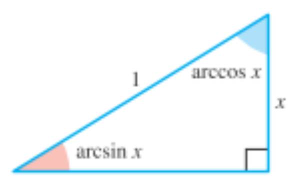
As we can see from Figure, the arccosine of x satisfies the identity:

$\cos^{-1} x + \cos^{-1}(-x) = \pi$



we can see from the triangle in Figure that for x > 0,

$\sin^{-1} x + \cos^{-1} x = \pi/2$



Inverses of $tan\ x$, $cot\ x$, $sec\ x$, and $csc\ x$

$$y = \tan^{-1} x \text{ is the number in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ for which } \tan y = x$$

$$y = \cot^{-1} x \text{ is the number in } [0, \pi] \text{ for which } \cot y = x$$

$$y = \sec^{-1} x \text{ is the number in } \left[0, \frac{\pi}{2}\right) \text{ U } \left(\frac{\pi}{2}, \pi\right] \text{ for which } \sec y = x$$

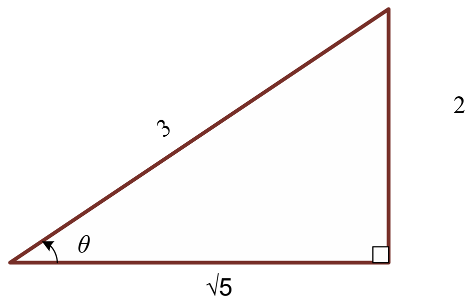
$$y = \csc^{-1} x \text{ is the number in } \left[-\frac{\pi}{2}, 0\right) \text{ U } \left(0, \frac{\pi}{2}\right] \text{ for which } \csc y = x$$

Example 2: Find $\cos\theta$, $\tan\theta$, $\sec\theta$ and $\csc\theta$ if

$$\theta = \sin^{-1}\frac{2}{3}$$

Solution:

$$\cos\theta = \frac{\sqrt{5}}{3}$$
$$\tan\theta = \frac{2}{\sqrt{5}}$$
$$\sec\theta = \frac{3}{\sqrt{5}}$$
$$\csc\theta = \frac{2}{\sqrt{5}}$$



The Derivative of the Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}(u)) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u| < 1$$

$$\frac{d}{dx}(\cos^{-1}(u)) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad |u| < 1$$

$$\frac{d}{dx}(\tan^{-1}(u)) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\cot^{-1}(u)) = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}(\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

$$\frac{d}{dx}(\csc^{-1}(u)) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

Example 3: Find ***y'*** for the following equations:

a) $y = \tan^{-1}(\sin x) \Rightarrow y' = \frac{1}{1+\sin^2 x}.\cos x$

b) $y = \sin^{-1}(\sin x) \Rightarrow y' = \frac{1}{\sqrt{1-\sin^2 x}}.\cos x = \frac{1}{\cos x}.\cos x = 1$

c)

$$y = \frac{x.\tan^{-1}(\cos x)}{\sqrt{x}} = \frac{\sqrt{x}.\cancel{\sqrt{x}}.\tan^{-1}(\cos x)}{\cancel{\sqrt{x}}}$$

$$y' = \sqrt{x}.\tan^{-1}(\cos x) = \sqrt{x}.\frac{1}{1+\cos^2 x}.\sin x + \tan^{-1}(\cos x).\frac{1}{2\sqrt{x}}$$

d) $y = \sec^{-1}(\csc x) \Rightarrow y' = \frac{-1}{\cancel{\csc x}\sqrt{\csc^2 x - 1}}.\cancel{\csc x}.\cot x = \frac{-\cot x}{\sqrt{\cot^2 x}} = -1$

Example 4: Find ***y'*** for the following equation:

$$y = \cot^{-1}(\sqrt{x}).\frac{\sec^2 x - 1}{\sin x}.\frac{1}{\tan^2 x.\csc x}$$

$$y = \cot^{-1}(\sqrt{x}).\frac{\cancel{\tan^2 x}}{\cancel{\sin x}}.\frac{1}{\cancel{\tan^2 x}.\frac{\cancel{1}}{\cancel{\sin x}}} = \cot^{-1}(\sqrt{x})$$

$$y' = \frac{-1}{1+(\sqrt{x})^2}.\frac{1}{2(\sqrt{x})} = \frac{-1}{2\sqrt{x}(1+x)}$$

Integration Formulas of the Inverse Trigonometric Functions

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \ldots\ldots\ldots \text{valid for } u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \ldots\ldots\ldots \text{valid for all u}$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \ldots\ldots\ldots \text{valid for} |u| > a > 0$$

Example 5: Evaluate the following integrals:

a) .

$$\int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big]_{1/\sqrt{2}}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

b) .

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

c) .

$$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x)\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \sec^{-1}(\sqrt{2}) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

d) .

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

e) .

$$\int \frac{dx}{\sqrt{3-4x^2}} = \int \frac{dx}{\sqrt{\left(\sqrt{3}\right)^2 - (2x)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

Example 6: Evaluate the following integral:

$$\int \frac{dx}{\sqrt{4x-x^2}}$$

Solution: The expression does not match any of the formulas, so we need to rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2$$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{a^2-u^2}}$$

$$\sin^{-1}\left(\frac{u}{a}\right) + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Homework

1. Evaluate the following integrals:
   a) .
$$\int \frac{dx}{17 + x^2}$$
   b) .
$$\frac{dx}{x\sqrt{5x^2 - 4}}$$
   c) .
$$\int \frac{dx}{2 + (x - 1)^2}$$
   d) .
$$\int_{-\pi/2}^{\pi/2} \frac{2\cos x\, dx}{1 + (\sin x)^2}$$
   e) .
$$\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$$
   f) .
$$\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$$
   g) .
$$\int \frac{dx}{\sin^{-1} x\,\sqrt{1 - x^2}}$$
   h) .
$$\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$$

2. Find the derivatives of the following equations:
$$y = \sin^{-1}(1 - t)$$
$$y = \csc^{-1}(x/2)$$
$$y = \ln(\tan^{-1} x)$$
$$y = \cot^{-1}(1/x) - \tan^{-1} x$$
$$y = \ln(x^2 + 4) - x\tan^{-1}(x/2)$$