Subject Name: Calculus I2

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Lecture No. 3

Lecture Title: The Exponential Function

The number *e*

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

The number e can be calculated as the limit

Its value is calculated with a computer to 15 places accuracy

e = **2.718281828459045**.

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

We have not yet found a way to give an obvious meaning to ex for x irrational.
But ln-1 x has meaning for any x, rational or irrational.
So Equation (1) provides a way to extend the definition of ex to irrational values of x.
The function ln-1 x is defined for all x, so we use it
to assign a value to ex at every point where had no previous definitio

$e^2 = e.e\ ,\quad e^{-2} = \frac{1}{e^2}\ ,\quad e^{1/2} = \sqrt{e}$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r.1 = r$$

Since $\ln x$ is one-to-one and

$$\ln(\ln^{-1} r) = r$$

this equation tells us that

$e^r = \ln^{-1} r = \exp r$ for r rational

Generally

For every real number x,

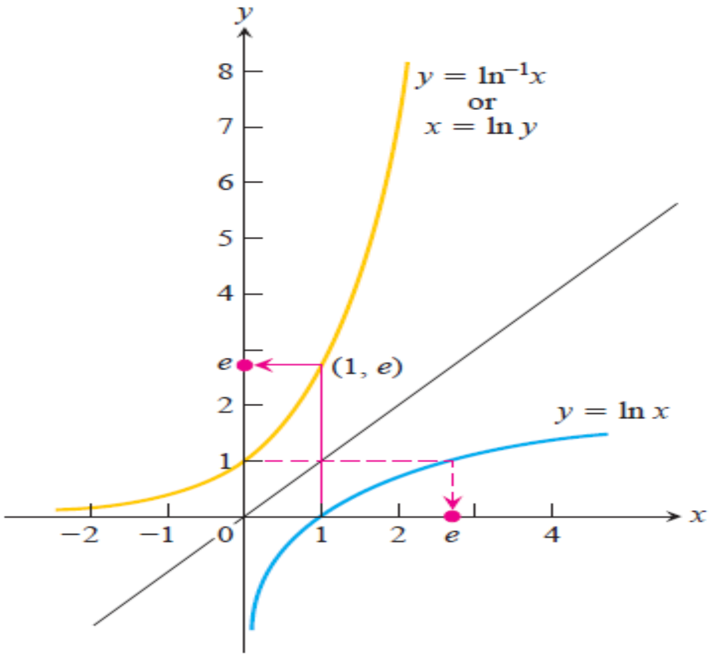
$$\mathbf{e^x = \ln^{-1} x = \exp x.}$$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \quad for\ all\ x > 0$$

$$\ln(e^x) = x \qquad for\ all\ \ x$$

The domain of $\ln x$ is $(0,\infty)$, and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$, and its range is $(0, \infty)$.



Example Using the Inverse Equations

$\ln e^2 = 2$

$\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$

$\ln e^{\sin x} = \sin x$

$e^{\ln 2} = 2$

$e^{\ln(x^2+1)} = x^2 + 1$

$e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$

Example

Find k if $e^{2k}=10$

Sol.

Take the natural logarithm of both sides:

$\ln e^{2k} = \ln 10$

$2k = \ln 10$

$$k = \frac{1}{2}\ln 10$$

Laws of Exponents for e^x

For all numbers x, x_1, and x_2 the natural exponential e^x obeys the following laws:

1. $e^{x_1} . e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Example

Applying the Exponent Laws

1. $e^{x+\ln 2} = e^x . e^{\ln 2} = 2e^x$

2. $e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

3. $\frac{e^{2x}}{e} = e^{2x-1}$

4. $(e^3)^x = e^{3x} = (e^x)^3$

The Derivative of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\frac{d}{dx}e^x = e^x$$

If u is any differentiable function of x, then

$$\boldsymbol{\frac{d}{dx}e^u = e^u\frac{du}{dx}}$$

Example

Find dy/dx for the following

1. $y = 5e^x$

Sol.

$$\frac{dy}{dx} = 5e^x$$

2. $y = e^{\sin x}$

Sol.

$$\frac{dy}{dx} = e^{\sin x}.\frac{d}{dx}(\sin x) = \cos x\,.\,e^{\sin x}$$

a^x and $\log_a x$

For any numbers $a > 0$ and x, the exponential function with base a is:

$$\boldsymbol{a^x = e^{x \ln a}}$$

The Derivative of a^x

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\boldsymbol{\frac{d}{dx} a^u = a^u . \ln a \frac{du}{dx}}$$

Example

Find dy/dx for the following

1. $y = 3^x$

Sol.

$$\frac{dy}{dx} = 3^x . \ln 3$$

2. $y = 3^{-x}$

Sol.

$$\frac{dy}{dx} = 3^{-x} . \ln 3 \frac{d}{dx}(-x) = -3^{-x} . \ln 3$$

3. $y = 3^{\sin x}$

Sol.

$$\frac{dy}{dx} = 3^{\sin x}.\ln 3\frac{d}{dx}(\sin x) = 3^{\sin x}.\ln 3.\cos x$$

The Inverse of a^x

For any positive number $\neq 1$,

$\log_a x$ **is the inverse function of** a^x.

$$a^{\log_a x} = x \qquad \text{for } x > 0$$

$$\log_a (a^x) = x \qquad \text{for all } x$$

And

$$\log_a x = \frac{\ln x}{\ln a}$$

The Derivative of $\log_a x$

To find derivatives involving base *a* logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}\log_a u = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}.\frac{d}{dx}\ln u = \frac{1}{\ln a}.\frac{1}{u}.\frac{du}{dx}$$

$$\boldsymbol{\frac{d}{dx}\log_a u = \frac{1}{\ln a}.\frac{1}{u}.\frac{du}{dx}}$$

Example

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}.\frac{1}{3x+1}.\frac{d}{dx}(3x+1) = \frac{3}{\ln 10}.\frac{1}{3x+1}$$