Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

Subject Name: Mathmatics II 2

2nd Class, Second Semester

Subject Code: [UOMU011042]

Academic Year: 2024-2025

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Lecture No.: (6-2)

Lecture Title: [Differentiation of Power Series]

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Differentiation of Power Series

A function defined by a power series has derivatives of all orders at every point of the interior of its interval of convergence. The first derivative is obtained by

$$\frac{d}{dx}\sum_{n=0}^{\infty}(a_n x^n) = \sum_{n=0}^{\infty}(n a_n x^{n-1})$$

For the second derivatives, the terms are differentiated again, and so on.

The Term by Term Differentiation Theorem;

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence ρ, then

1. $\sum_{n=0}^{\infty}(n a_n x^{n-1})$ also has radius of convergence ρ.

2. $f(x)$ is differentiable on $(-\rho, \rho)$

3. $f'(x) = \sum_{n=0}^{\infty}(n a_n x^{n-1})$ on $(-\rho, \rho)$

Examples:

1. Prove that $\frac{d}{dx}(sinx) = cosx$.

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{d}{dx}(sinx) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = cosx$$

Note that convergence at one or both endpoints of the interval of convergence of a power series may be lost in the process of differentiation.

2. **The series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$.**

The series of derivatives

$$f'(x) = \sum_{n=0}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots$$

Is a geometric series that converges only for $-1 < x < 1$. The series diverges at the endpoints $x = \pm 1$.

Integration of Power Series:

Just as a power series may be differentiated term by term, it may also be integrated term by term. The new series will surely converge in the open interval where the orginal series converges, and it may converge at one or both of the endpoints as well.

The Term by Term Integration Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence ρ, then

1. $\sum_{n=0}^{\infty} \left(\frac{a_n x^{n+1}}{n+1}\right)$ **also has radius of convergence ρ.**
2. $\int f(x)dx$ **exists for x in $(-\rho, \rho)$**
3. $\int f(x)dx = \sum_{n=0}^{\infty} \left(\frac{a_n x^{n+1}}{n+1}\right)$ **on $(-\rho, \rho)$**

Examples:

1. **Find $ln(1 + x)$**

The series $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$

$$ln(1+x)=\int_0^x \frac{1}{1+t}dt=\int_0^x (1-t+t^2-t^3+\cdots)dt$$

$$=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots\Big|_0^x=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots,-1<x\le 1$$

Indeteminate Forms

We may determine the following form

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}$$

using Taylor series for one or more functions.

Examples:

1. Evaluate $\lim_{x\to 0}\frac{lnx}{(x-1)}$

Let $f(x)=lnx, f(1)=0$

$f'(x)=\frac{1}{x},\ f'(1)=1$

$f''(x)=-\frac{1}{x^2}\ ,f''(0)=-1$

So

$$Lnx=0+(x-1)-\frac{1}{2}(x-1)^2+\cdots,$$

$$\frac{Lnx}{x-1}=1-\frac{1}{2}(x-1)+\cdots,$$

and

$$\lim_{x\to 1}\frac{\ln x}{(x-1)} = \lim_{x\to 0}\left(1-\frac{1}{2}(x-1)+\cdots\right) = 1$$

2. $\lim_{x\to 0}\frac{\sin x - \tan x}{x^3}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\tan x = x + \frac{x^3}{3!} + \frac{12}{5}x^5 + \cdots$$

$\sin x - \tan x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \left(x + \frac{x^3}{3} + \frac{12}{5}x^5 + \cdots\right)$$

$$= -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$\lim_{x\to 0}(\sin x - \tan x) = -\frac{1}{2}$$

3. $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$$

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x\sin x} = \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x.\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \frac{x^3\left(\frac{1}{3!}+\frac{x^2}{5!}-\cdots\right)}{x^2.\left(1-\frac{x^2}{3!}+\cdots\right)} = x.\frac{\frac{1}{3!}+\frac{x^2}{5!}-\cdots}{1-\frac{x^2}{3!}+\cdots}$$

Therefor

$$\lim_{x\to 0}\left(\frac{1}{sinx} - \frac{1}{x}\right) = \lim_{x\to 0}\left(x.\frac{\frac{1}{3!}+\frac{x^2}{5!}-\cdots}{1-\frac{x^2}{3!}+\cdots}\right) = 0$$