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Lecture No. 7

Lecture Title: Integration by Parts

8.2

Integration by Parts

Since

$$\int x \, dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2 \, dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x) \, dx \text{ is not equal to } \int f(x) \, dx \cdot \int g(x) \, dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x) \, dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x \, dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x \, dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x , the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)] \, dx = \int [f'(x)g(x) + f(x)g'(x)] \, dx$$

or

$$\int \frac{d}{dx} [f(x)g(x)] dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x) dx = \int \frac{d}{dx} [f(x)g(x)] dx - \int f'(x)g(x) dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \quad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x) dx$ and $dv = g'(x) dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u dv = uv - \int v du \quad (2)$$

This formula expresses one integral, $\int u dv$, in terms of a second integral, $\int v du$. With a proper choice of u and v , the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv . The next examples illustrate the technique.

EXAMPLE 1 Using Integration by Parts

Find

$$\int x \cos x dx.$$

Solution We use the formula $\int u dv = uv - \int v du$ with

$$\begin{aligned} u &= x, & dv &= \cos x dx, \\ du &= dx, & v &= \sin x. \end{aligned} \quad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C. \quad \blacksquare$$

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x \cos x dx = \int u dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v .

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x) \, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x) \, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx , as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$$\begin{aligned} u &= \ln x && \text{Simplifies when differentiated} && dv &= dx && \text{Easy to integrate} \\ du &= \frac{1}{x} \, dx, && && v &= x. && \text{Simplest antiderivative} \end{aligned}$$

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C. \quad \text{■}$$

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x dx.$$

Solution With $u = x^2$, $dv = e^x dx$, $du = 2x dx$, and $v = e^x$, we have

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We say more about this later in this section when we discuss *tabular integration*.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x dx.$$

Solution Let $u = e^x$ and $dv = \cos x dx$. Then $du = e^x dx$, $v = \sin x$, and

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x dx, \quad v = -\cos x, \quad du = e^x dx.$$

Then

$$\begin{aligned} \int e^x \cos x dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \quad \blacksquare$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x) \, dx = f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) \, dx \quad (3)$$

In applying Equation (3), we normally use the u and v notation from Equation (2) because it is easier to remember. Here is an example.

EXAMPLE 6 Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x -axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x} \, dx.$$

Let $u = x$, $dv = e^{-x} \, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned} \int_0^4 xe^{-x} \, dx &= -xe^{-x} \Big|_0^4 - \int_0^4 (-e^{-x}) \, dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x} \, dx \\ &= -4e^{-4} - e^{-x} \Big|_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91. \quad \blacksquare \end{aligned}$$

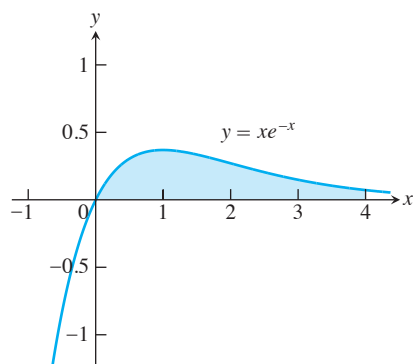


FIGURE 8.1 The region in Example 6.

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x) \, dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize

the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

Compare this with the result in Example 4.

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x) dx.$$

(Remember that g may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u dv$$

by choosing dv to be part of the integrand including dx and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula

$$\int u dv = uv - \int v du.$$

If the new integral on the right side is more complex than the original one, try a different choice for u and dv .

EXAMPLE 9 A Reduction Formula

Obtain a “reduction” formula that expresses the integral

$$\int \cos^n x dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x dx,$$

so that

$$du = (n-1)\cos^{n-2} x (-\sin x dx) \quad \text{and} \quad v = \sin x.$$

Hence

$$\begin{aligned} \int \cos^n x dx &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x dx \\ &= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx, \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx. \end{aligned}$$

If we add

$$(n-1) \int \cos^n x dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx.$$

We then divide through by n , and the final result is

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x \, dx = \sin x + C \quad \text{or} \quad \int \cos^0 x \, dx = \int dx = x + C. \quad \blacksquare$$

EXAMPLE 10 Using a Reduction Formula

Evaluate

$$\int \cos^3 x \, dx.$$

Solution From the result in Example 9,

$$\begin{aligned} \int \cos^3 x \, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned} \quad \blacksquare$$

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

- | | | | |
|---------------------------------|--|---|--|
| 1. $\int x \sin \frac{x}{2} dx$ | 2. $\int \theta \cos \pi \theta d\theta$ | 13. $\int (x^2 - 5x)e^x dx$ | 14. $\int (r^2 + r + 1)e^r dr$ |
| 3. $\int t^2 \cos t dt$ | 4. $\int x^2 \sin x dx$ | 15. $\int x^5 e^x dx$ | 16. $\int t^2 e^{4t} dt$ |
| 5. $\int_1^2 x \ln x dx$ | 6. $\int_1^e x^3 \ln x dx$ | 17. $\int_0^{\pi/2} \theta^2 \sin \theta d\theta$ | 18. $\int_0^{\pi/2} x^3 \cos 2x dx$ |
| 7. $\int \tan^{-1} y dy$ | 8. $\int \sin^{-1} y dy$ | 19. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t dt$ | 20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2) dx$ |
| 9. $\int x \sec^2 x dx$ | 10. $\int 4x \sec^2 2x dx$ | 21. $\int e^\theta \sin \theta d\theta$ | 22. $\int e^{-y} \cos y dy$ |
| 11. $\int x^3 e^x dx$ | 12. $\int p^4 e^{-p} dp$ | 23. $\int e^{2x} \cos 3x dx$ | 24. $\int e^{-2x} \sin 2x dx$ |

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}} ds$

26. $\int_0^1 x\sqrt{1-x} dx$

27. $\int_0^{\pi/3} x \tan^2 x dx$

28. $\int \ln(x + x^2) dx$

29. $\int \sin(\ln x) dx$

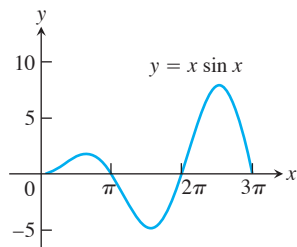
30. $\int z(\ln z)^2 dz$

Theory and Examples

31. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x -axis (see the accompanying figure) for

a. $0 \leq x \leq \pi$ b. $\pi \leq x \leq 2\pi$ c. $2\pi \leq x \leq 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x -axis for $n\pi \leq x \leq (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x -axis (see the accompanying figure) for

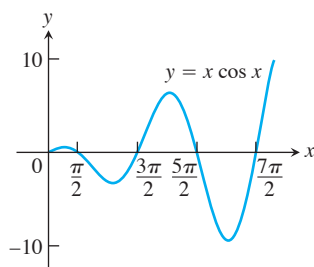
a. $\pi/2 \leq x \leq 3\pi/2$ b. $3\pi/2 \leq x \leq 5\pi/2$

c. $5\pi/2 \leq x \leq 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x -axis for

$$\left(\frac{2n-1}{2}\right)\pi \leq x \leq \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y -axis. b. about the line $x = 1$.

35. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \leq x \leq \pi/2$, about

a. the y -axis. b. the line $x = \pi/2$.

36. Finding volume Find the volume of the solid generated by revolving the region bounded by the x -axis and the curve $y = x \sin x$, $0 \leq x \leq \pi$, about

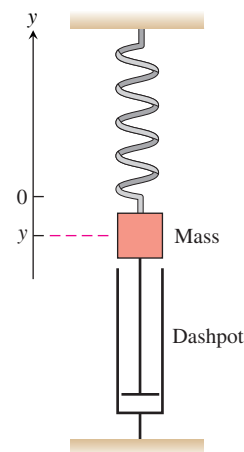
a. the y -axis. b. the line $x = \pi$.

(See Exercise 31 for a graph.)

37. Average value A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.



38. Average value In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t} (\sin t - \cos t), \quad t \geq 0.$$

Find the average value of y over the interval $0 \leq t \leq 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

$$39. \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$40. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$41. \int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx, \quad a \neq 0$$

$$42. \int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x) \, dx &= \int y f'(y) \, dy && \begin{array}{l} y = f^{-1}(x), \quad x = f(y) \\ dx = f'(y) \, dy \end{array} \\ &= y f(y) - \int f(y) \, dy && \begin{array}{l} \text{Integration by parts with} \\ u = y, \, dv = f'(y) \, dy \end{array} \\ &= x f^{-1}(x) - \int f(y) \, dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x \, dx &= \int y e^y \, dy && \begin{array}{l} y = \ln x, \quad x = e^y \\ dx = e^y \, dy \end{array} \\ &= y e^y - e^y + C \\ &= x \ln x - x + C. \end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned} \int \cos^{-1} x \, dx &= x \cos^{-1} x - \int \cos y \, dy && y = \cos^{-1} x \\ &= x \cos^{-1} x - \sin y + C \\ &= x \cos^{-1} x - \sin(\cos^{-1} x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - \int f(y) \, dy \quad y = f^{-1}(x) \quad (4)$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x .

$$43. \int \sin^{-1} x \, dx$$

$$44. \int \tan^{-1} x \, dx$$

$$45. \int \sec^{-1} x \, dx$$

$$46. \int \log_2 x \, dx$$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx. \quad (5)$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

$$\text{a. } \int \cos^{-1} x \, dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

$$\text{a. } \int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C \quad \text{Eq. (4)}$$

$$\text{b. } \int \tan^{-1} x \, dx = x \tan^{-1} x - \ln \sqrt{1 + x^2} + C \quad \text{Eq. (5)}$$

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with (a) Eq. (4) and (b) Eq. (5). In each case, check your work by differentiating your answer with respect to x .

$$49. \int \sinh^{-1} x \, dx$$

$$50. \int \tanh^{-1} x \, dx$$